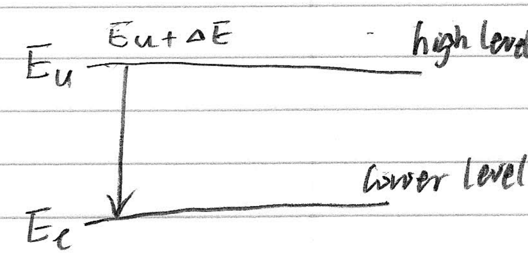
Dec. 11, 2012 ⑨

Line profiles (CH 4.8)

* The Lorentz Profile (CH 4.8.1)
* The Doppler Broadening profile (CH 4.8.2)

A spectral line has a natural width $\Delta\nu$ around the line center frequency ν_0

* spectral line is caused by bound-bound transition of electrons between different energy level



$$h\nu_0 = E_u - E_l$$

ν_0: photon frequency at the line center.

* Electrons at a high energy level has a finite time

e.g. Hydrogen $2p \to 1s$; $\tau = 1.6\times10^{-9}\,s$

$\tau \to \Delta E$

* Heisenberg uncertainty principle

$$\Delta E\,\Delta t = \hbar \;;\; \hbar = \frac{h}{2\pi},\; h = 6.626\times10^{-27}\ \text{ergs}$$

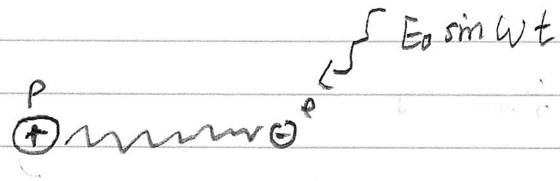
Planck's constant

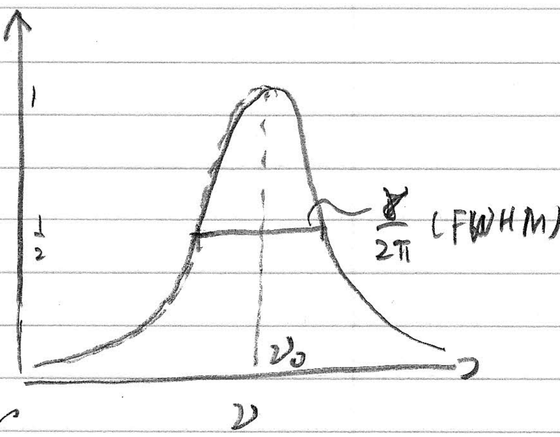
ΔE is finite, not zero

$\Rightarrow \Delta\nu = \dfrac{\Delta E}{h} \Rightarrow$ the natural width of spectral line

Lorentz profile : the frequency profile of spectrum line

* Use "classical oscillator" approach to derive the profile σ





the atom is treated as an oscillator that undergoes a stimulated oscillation, for a given external electron field $E = E_0 \sin \omega t$

caused by incident photons.

Equation of motion of the bound electron

$$m_e \frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = -e E_0 e^{i\omega t}$$

γ : damping constant caused by radiation

The solution is $z = z_0 e^{i\omega t}$

$$z_0 = \frac{e}{m_e} \frac{E_0}{\omega_0^2 - \omega^2 + i\gamma\omega}$$: amplitude of the oscillation

ω_0 : resonant frequency or natural frequency

At $\omega = \omega_0$, largest amplitude, or absorption

$\omega \neq \omega_0$. smaller amplitude, no absorption

This leads to the damping profile or Lorentz profile

The absorption cross section σ_a

$$\sigma_a = \frac{e^2}{mc} f \frac{(\gamma/4\pi)}{(\nu - \nu_0)^2 + (\frac{\gamma}{4\pi})^2} \quad \text{------ (4.73)}$$

f : all the quantum effects, not in the classical model

f: oscillation strength

$$f = \frac{mc^3}{2\pi e^2 \nu^2} \frac{g_u}{g_l} A_{ul}$$: from quantum mechanics.

A_{ul}: Einstein probability coefficient

g_u, g_l: statistical weight of upper, low levels

$$\gamma_u = 4\pi \sum_{l<u} A_{ul}$$

$$\gamma = \gamma_u + \gamma_l$$

γ: determine the natural width of spectral lines
but usually small, e.g. $\Delta\lambda = 10^{-4}$ Å

$$\boxed{\sigma_a n_a = \kappa \rho}$$

n_a: number density (number of absorber)
ρ: mass density
κ: mass opacity ($cm^2\, g^{-1}$)

$$FWHM = \frac{\gamma}{2\pi}$$

(Full width at half maximum)

Lorentz profile: narrow core ($\frac{\gamma}{2\pi}$), broad wing

$$\nu \gg \nu_0, \quad \sigma_a \propto \frac{1}{(\nu - \nu_0)^2} \neq 0$$

Doppler effect (CH 4.8.2)

— thermal broadening of spectral lines

The true width of a stellar spectral line is mainly caused by Doppler broadening, since the high stellar temperature $10^3 - 10^4$ K in the atmosphere.

* Doppler shift $\Delta\nu = \frac{v}{c}\nu_0$; $\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta\nu}{\nu_0} = \frac{v}{c}$

v: velocity of atoms

$v_0^2 = \frac{2kT}{m} \Rightarrow \boxed{\Delta\nu_D = \frac{v_0}{c}\nu_0}$ Doppler width

* Maxwell - Boltzmann Distribution of thermal particles

$$\frac{dn(v,\mu)}{n} = (2\pi)^{-\frac{1}{2}}\left(\frac{m}{kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{kT}} v^2 dv\, d\mu$$

$\mu = \cos\theta$

To obtain true spectral profile, one needs to convolve the Lorentz profile with Maxwell Eq.

$$\sigma_a(\nu,\nu_0,T) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left(\frac{m}{kT}\right)^{\frac{3}{2}} \frac{e^2}{mc} f \frac{\gamma}{4\pi}\int_0^\infty\int_{-1}^{+1} \frac{e^{-\frac{mv^2}{2kT}} v^2 dv\, d\mu}{(\nu-\nu_0-\nu_0\frac{v\mu}{c})^2 + (\frac{\gamma}{4\pi})^2}$$

$$\sigma_a(\nu,\nu_0,T) = \frac{e^2 f}{mc}\pi^{\frac{1}{2}}\frac{1}{\Delta\nu_D} H\left(a, \frac{\Delta\nu}{\Delta\nu_D}\right) \quad \text{---} \quad (4.77)$$

a: $a = \frac{\gamma}{4\pi}\frac{1}{\Delta\nu_D}$: ratio of natural width and Doppler width

$a \sim 10^{-3}$ typical value

$\Delta\nu_D \sim 10^{-1}$ Å

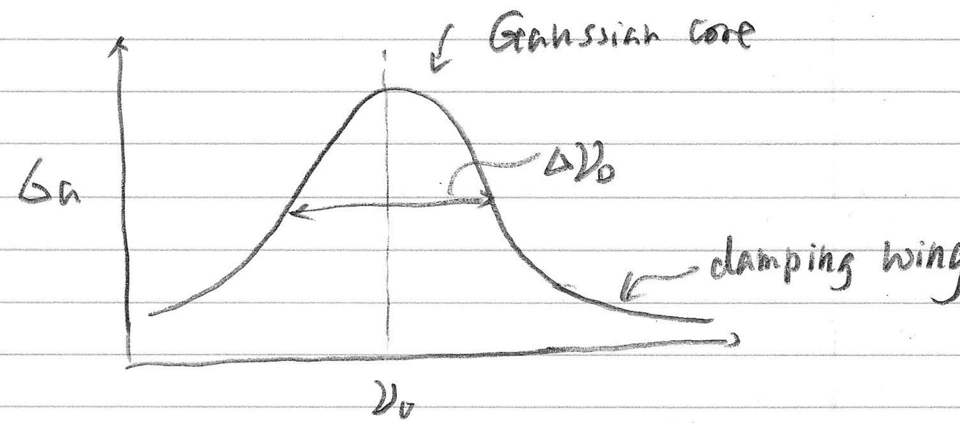
$\Delta\nu = \nu - \nu_0$: distance to line center

$$H(a, u = \frac{\Delta\nu}{\Delta\nu_D}) = \frac{a}{\pi}\int_{-\infty}^{+\infty} \frac{e^{-y^2}dy}{a^2+(u-y)^2} \quad \text{— (4.81)}$$

The Voigt function

$u \sim 1$, $H \sim e^{-u^2}$: Gaussian core

$u \gg 1$, $H \sim \frac{1}{u^2}$: broad damping wing



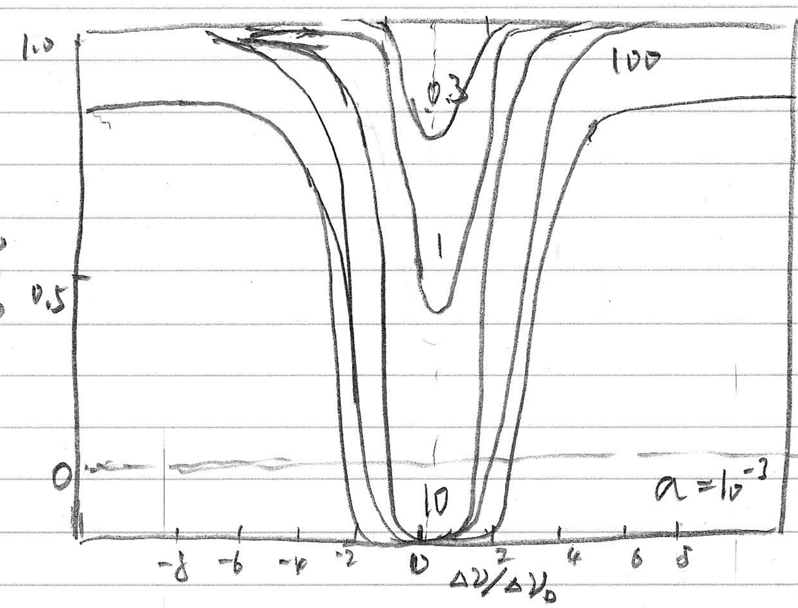
Curve of Growth (CH 4.8.3)

— The evolution of the intensity of spectral lines with increasing number density of absorpers

$$\frac{I_\nu}{I_0} = e^{-\tau_\nu} = e^{-\beta_0} = e^{-n_a \sigma_a \Delta r} \quad \cdots (4.82)$$

$\beta_0 \ll 1$, weak absorption line

$\beta_0 \uparrow$, stronger absorption line, depending on n_a, σ_a



$\beta_0 = 0.3$

$\beta_0 = 1$ — Linear

$\beta_0 = 10$

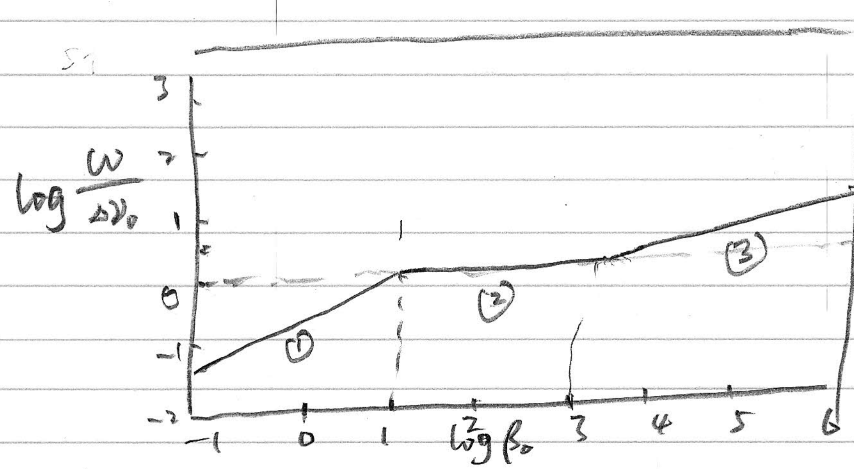
$\beta_0 = 100$

— saturation

$\beta_0 = 10^4$

damping wing

Fig. 4.10 (P230)



$W = \nu_D$

Curve of growth

Fig 4.11 (P231)

* Curve of growth has three segments

① Linear region: ($W \sim \beta_0$)

$\tau < 1$, $\log \beta_0 < 0$, $W < \Delta\nu_D$

optical thin, small β, weak line

② Saturation region ($W \sim \sqrt{\ln \beta_0} \sim$ constant)

$1 < \tau < 1000$, $0 < \log\beta_0 < 3$, $W \sim \Delta\nu_D$

Doppler Gaussian core is saturated, the line width does not grow.

$W \sim \Delta\nu_D$

The equivalent width doesn't increase with increasing β_0, or absorber

③ Damping region ($W \sim \beta_0^{\frac{1}{2}}$)

$\tau > 10^3$, $\log\beta > 3$, $W > \Delta\nu_D$

Lorentz damping wing participates in the absorption

The line width starts to grow again, but not as steep as in linear region

The curve of growth can be used to determine accurately the temperature and abundance.

For instance, Fe has many lines with different β_0.
measuring W leads to number of Fe (β) and T ($\Delta\nu_D$)