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Stellar modeling (CH7)

* putting together all the physics, find the realistic solution ① analytic (ideally)
  ② numerically (real, hopefully)
* "All the physics" means ① a set of differential eqs.
  ② a set of microscopic constituent eqs

Let's re-visit "All the physics" again
The equations of stellar structure

① Mass Eq $\frac{dM_r}{dr} = 4\pi r^2 \rho \Rightarrow M_r(r)$ (7.6)

② Momentum Eq. $\frac{dP}{dr} = -\rho g \Rightarrow P(r)$ (7.5)

③ Energy Eq. $\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon \Rightarrow L(r)$ (7.7)

This energy equation is incomplete. we need $T(r)$

④ Energy Eq (The energy diffusion equation)

$$\nabla = \frac{d\ln T}{d\ln P} = -\frac{r^2 P}{GM_r \rho}\frac{1}{T}\frac{dT}{dr} \Rightarrow T(r)$$

If radiative zone, $\nabla = \nabla_{rad} = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{L(r)}{GM_r}$
$(\nabla > \nabla_{ad})$

If convection zone $\nabla = \nabla_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2}$
$(\nabla < \nabla_{ad})$

Thus, there are $\boxed{\text{FOUR}}$ differential equation

The microscopic constituent physics equations

⑤ $P = P(\rho, T, X)$ $\Rightarrow \rho(r)$ --- (7.1)

X: composition

⑥ $E = E(\rho, T, X)$ $\Rightarrow E(r)$ --- (7.2)

⑦ $\kappa = \kappa(\rho, T, X)$ $\Rightarrow \kappa(r)$ --- (7.3)

⑧ $\varepsilon = \varepsilon(\rho, T, X)$ $\Rightarrow \varepsilon(r)$ --- (7.4)

These "Four" equations are functions, only depending on local ρ, T and X.

They can be pre-calculated and tabulated

Therefore, there are EIGHT equations in total:

Four differential + Four functions

$\Rightarrow M_r, P, L, T, \rho, E, \kappa, \varepsilon$

$\Rightarrow$ a complete set of equations, physics-based

Numerical solution (CH7.2.2)

* Solving four coupled first-order ordinary differential equations.

* It is not difficult in modern ages, given
  1. advanced numeric method, 2. advanced computer.

* Need to specify bounary conditions.

Inner boundary $r=0$:

$$M_r = 0$$

$$\frac{dP}{dr} = 0$$

$$L = 0$$

$$\frac{dT}{dr} = 0$$

Outer boundary, $r=R$:

$$M_r = M$$

$$P = 0$$

$$L = L_{Total}$$

$$T = 0$$

This is simplified approach.

Numerical Methods (CH 7.2.2)

1. Marching Integration Method.

   Initial value problem: integrate forward from the center, untic reach the surface $P=0$.
   Use "Runge-kutta" method to integrate

2. Finite difference method. (standard iteraction method)

   Also called "Newton-Raphson" method.
   Solving equations on a pre-constructed grid.
   Construct a set of linear equations → solving big matrix

Polytropes and Polytropic Equations of state

→ Find a realistic analytic solution to the set of equations

— of course, it is constrained by the assumption

* Constant density model $\rho = \rho_c$

Four coupled differential equations are reduced to single differential equation : $\frac{dP}{dr} = -\rho_c g$.

One can find $P(r)$, $\rho(r)$, $M(r)$, $T(r)$

* Linear density model $\rho = \rho_c (1 - \frac{r}{R})$

Again, four differential equations are reduced to one equation $\Rightarrow P(r), \rho(r), M(r), T(r)$

* To be more realistic, introduce polytrope

$$P(r) = K \rho(r)^{1+\frac{1}{n}}$$

n: polytropic index

K: constant

e.g. degenerate non-relativistic electrons in white dwarfs.

$$P_e = 1.004 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}} \text{ dyne cm}^{-2} \quad (7.14)$$

$$\Rightarrow 1 + \frac{1}{n} = \frac{5}{3} \Rightarrow n = \frac{3}{2} = 1.5$$

e.g. Adiabatic convection zone

$$P = K \rho^{\Gamma_2}$$

For ideal gas, $\Gamma_2 = \frac{5}{3} \Rightarrow n = \frac{3}{2}$

"Lane-Emden" Equation

The polytropic assumption lead to "Lane-Emden" eq.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n \quad \text{------} \ (7\text{-}26)$$

* The four differential equations are reduced to one second-order normal differential equation.

* The dimensionless parameters.

$$\xi = \frac{r}{r_n} : \text{ dimensionless } r \quad \text{------} \ (7.24)$$

$$r_n^2 = \frac{(n+1)P_c}{4\pi G \rho_c^2} \quad \text{------} \ (7\text{-}25)$$

P_c, ρ_c = the values at center $(r=0)$

θ : dimensionless ρ

$$\rho(r) = \rho_c\,\theta^n(r) \quad \text{------} \ (7\text{-}20)$$

$r=0$, $\theta=1$, $\rho(0)=\rho_c$, $\xi=0$ at center

$r=R$, $\theta=0$ $\rho(R)=0$, $\xi=\xi_1$ at surface

Analytic solution available for $n=0, 1, 5$,

* $n=0$, $\rho=\rho_c$ constant density model

$$\theta_0(\xi) = 1-\frac{\xi^2}{6}, \quad \xi_1=\sqrt{6}$$

$$P = P_c\left(1-\frac{r}{R}\right)^2$$

* $n=1$, $\theta_1(\xi) = \dfrac{\sin\xi}{\xi}$, $\xi_1=\pi$

* $n=5$, $\theta_5(\xi) = \left[1+\dfrac{\xi^2}{3}\right]^{-\frac{1}{2}}$, with $\xi_1 \to \infty$