Nov. 20, 2012

Equation of Motion (CH 5.1.3)

Use buoyancy force to drive the convection

$$\rho \frac{dw}{dt} = (\rho - \rho') g \quad \text{--- (5.22)}$$

$$\frac{dw}{dt} = \frac{\rho - \rho'}{\rho} g$$

We need to relate $\rho - \rho' \Rightarrow T - T'$,

Use the coefficient of thermal expansion Q.

$$Q = -\left(\frac{d \ln \rho}{d \ln T}\right)_P \quad \text{density change with } T$$

$$\rho = \rho(T, P)$$

$$d \ln \rho = \frac{1}{\rho} d\rho = \frac{1}{\rho}(\rho - \rho')$$

$$d \ln T = \frac{1}{T} dT = \frac{1}{T}(T - T')$$

$$\Rightarrow \frac{\rho - \rho'}{\rho} = -Q \frac{T - T'}{T} = \frac{Q}{T} \Delta T \quad \text{--- (5.24)}$$

From definition of Q. $Q = \frac{\chi_T}{\chi_\rho}$ (dimensionless)

$Q = 1$ for ideal gas

$Q \neq 1$ for (1) mixture and (2) ionization

$$\frac{dw}{dt} \approx \frac{Q g}{T} \Delta T \quad \text{--- (5.26)}$$

Convective Efficiencies and Time Scales (CH 5.1.4)

① Energy $\frac{D\Delta T}{Dt} = (\beta - \beta_{ad}) w - \frac{\nu_T}{\ell^2}\Delta T$ → adiabatic, leaking

② momentum $\frac{dw}{dt} = \frac{Qg}{T}\Delta T$ → buoyancy

Find $\Delta T(t)$ and $w(t)$

$\frac{D}{Dt}$ of ① $\frac{D^2\Delta T}{Dt} = (\beta - \beta_{ad})\frac{dw}{dt} - \frac{\nu_T}{\ell^2}\frac{D\Delta T}{Dt}$

$$\frac{D^2\Delta T}{Dt} + \frac{\nu_T}{\ell^2}\frac{D\Delta T}{Dt} - \frac{Qg}{T}(\beta - \beta_{ad}) = 0$$

This is a classical damped oscillation equation

$$\begin{cases} \Delta T = \Delta T_0 e^{\sigma t} \\ w = w_0 e^{\sigma t} \end{cases}$$

⇒ σ: real positive : unstable
σ : real, negative : damping
σ : imaginary : oscillation

$$\Rightarrow \sigma^2 + \underbrace{\sigma\frac{\nu_T}{\ell^2}}_{\text{Leaking}} - \underbrace{\frac{Qg}{T}(\beta - \beta_{ad})}_{\text{buoyancy term adiabatic process}} = 0 \quad \text{---- } 5.27$$

Introduce $N^2 = -\frac{Qg}{T}(\beta - \beta_{ad})$ --- 5.28

N: Brunt - Väisälä frequency.

$$\sigma^2 + \sigma\frac{\nu_T}{\ell^2} + N^2 = 0$$

only consider the adiabatic buoyancy term

$$\sigma^2 = -N^2.$$

* If $\beta > \beta_{ad}$, $N^2 < 0$, $\sigma^2 > 0$. σ positive
  ΔT, w undergoes exponential growth ⇒ Convection

* If $\beta < \beta_{ad}$, $N^2 > 0$, $\sigma^2 < 0$ $\sigma = i\,\mathrm{Im}(\sigma)$, imaginary
  the parcel oscillates at the frequency N

* If consider the leaky term only ($\propto \frac{\nu^2}{\ell^2}$)

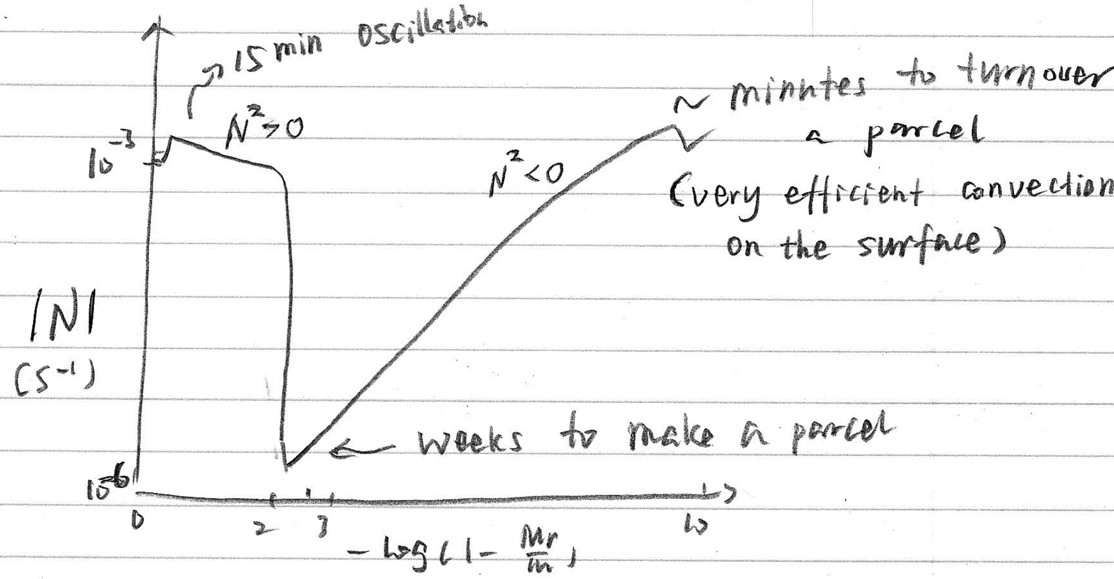
$$\sigma^2 + \sigma \frac{\nu_T}{\ell^2} = 0$$

$$\sigma = -\frac{\nu_T}{\ell^2} < 0 \quad \text{always negative}$$

Exponentially damping term.

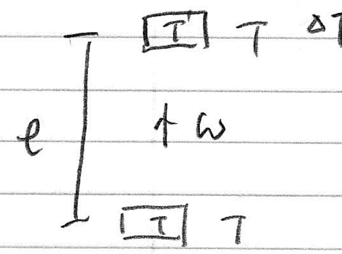
$0 \le \frac{\nu_T}{\ell^2} \ll 1$, slow radiative cooling, the case of efficient convection

$\frac{\nu_T}{\ell^2} \gg 1$. very efficient cooling, the parcel lose the identity quickly; less efficient convection



Sun Figure 5-3

* Convective Flux (CH 5.1.5)



$$F_{conv} = \rho C_p \Delta T \cdot w \qquad \text{erg cm}^{-2}\,\text{s}^{-1}$$

erg cm^{-3} cm/s

$$\boxed{w = \sigma \ell} \quad \text{---- (5.34)}$$

For adiabatic rising term only,

$$\frac{D\Delta T}{Dt} = (\beta - \beta_{ad})\, w$$

$$\frac{D\Delta T}{\Delta t} = \sigma \Delta T$$

$$\Delta T = \frac{(\beta - \beta_{ad})\, w}{\sigma} = (\beta - \beta_{ad})\, \ell$$

$$\sigma = \sqrt{-N^2} = \left[\frac{\alpha g}{T}(\beta - \beta_{ad})\right]^{\frac{1}{2}} \quad \text{from (5.28)}$$

$$\beta - \beta_{ad} = \frac{\sigma^2 T}{\alpha g}$$

$$\Rightarrow \boxed{\Delta T = \frac{\sigma^2 \ell T}{\alpha g}} \quad \text{----- (5.35)}$$

$$F_{conv} = \frac{\rho C_p T \sigma^3 \ell^2}{\alpha g} \quad \text{--- (5.36)}$$

$$L_{conv} = 4\pi r^2 F_{conv}$$