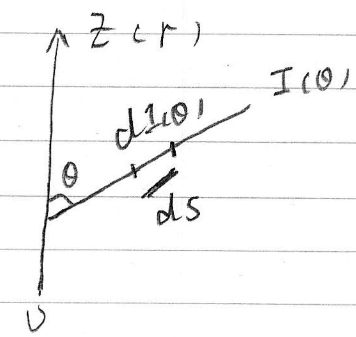
Nov. 6, 2012

* Radiation Transfer Equation

$$\frac{dI(\theta)}{ds} = \text{source} - \text{sink}$$



$$\frac{dI(\theta)}{ds} = \rho j - \kappa\rho I(\theta)$$

Change of radiation intensity along the path ds

$$\left(\frac{dI(\theta)}{ds}\right)_{source} = \rho j$$

j : mass emission coeffient due to direct emission from particles or scattering lights into θ direction

j : $erg\ s^{-1}\ g^{-1}$

$$\left(\frac{dI(\theta)}{ds}\right) = -\kappa\rho I(\theta)$$

κ : opacity ($\kappa I(\theta)$ = mass absorption) direct absorption of particles, or scattering by particles

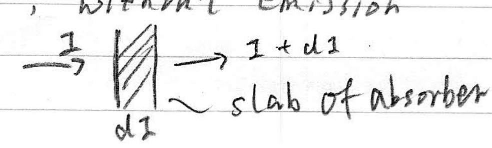
κ : $cm^2\ g^{-1}$

κρ : $cm^2\ g^{-1} \cdot g\ cm^{-3} = cm^{-1}$

$$\left(\frac{dI}{I}\right)_{sink} = -\kappa\rho\, ds \quad \text{(dimensionless)}$$

$\Rightarrow I = I_0 e^{-\kappa\rho s}$: exponential attenuation of the beam, without emission

$$I = I_0 e^{-\frac{s}{\lambda_f}}$$



$\lambda_f = \frac{1}{\kappa\rho}$: e-folding distance or mean free path of photons

* the total change

$$\frac{dI(\theta)}{ds} = \rho j - \kappa\rho I(\theta)$$

$$\frac{1}{\rho}\frac{dI(\theta)}{ds} = j - \kappa I(\theta) \quad \text{--- (4.4)}$$

Assuming complete $\left[\begin{matrix}\text{uniform}\\ \text{isotropic}\end{matrix}\right]$ and LTE

$$\frac{dI(\theta)}{ds} = 0$$

$$\Rightarrow \boxed{I(\theta) = \frac{j}{\kappa} = B(T)} \quad \text{--- (4.6)}$$

Standard Equation of Radiation Transfer

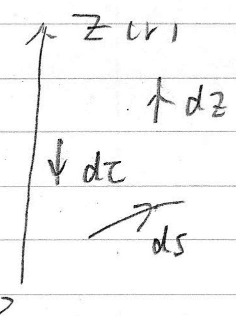
* Define $S_\nu = \frac{j_\nu}{\kappa_\nu}$: source function
frequency ν dependent

* Define optical depth (dimensionless depth)
(along the z direction)

$$d\tau = -\kappa\rho\, dz$$

$$d\tau = -\frac{dz}{\lambda_f} \; ; \quad dz = \cos\theta\, ds$$

the distance normalized by the photo mean free path and looking inward ($-z$) :



$z = 0$, $\tau = 0$: surface ; $\tau = 1$; $z = -\lambda$

$\tau < 1$: optical thin
$\tau > 1$ optical thick

$$\frac{dI_{(\theta)}}{ds} = \rho j - \rho \kappa I_{(\theta)}$$

$$\Rightarrow \mu\, d\tau = -\kappa\rho\, dz = -\kappa\rho\mu\, ds$$

$$ds = -\frac{1}{\kappa\rho\mu}\, d\tau$$

$$\Rightarrow \boxed{\mu \frac{dI_\nu(\tau,\mu)}{d\tau_\nu} = I_\nu(\tau,\mu) - S_\nu(\tau,\mu)} \quad \cdots (4.10)$$

This first-order normal differential equation can be solved analytically.

Solving the radiation transfer equation

Without loss of generality, dropping frequency ν

$$\mu \frac{dI}{d\tau} = I - S$$

Find $I(\tau, \mu)$: depends on τ and μ.

$$\frac{dI}{d\tau} - \frac{I}{\mu} = -\frac{S}{\mu}$$

multiply by $e^{-\frac{\tau}{\mu}}$: $e^{-\frac{\tau}{\mu}} \frac{dI}{d\tau} - e^{-\frac{\tau}{\mu}} \frac{I}{\mu} = \frac{d}{d\tau}\left(e^{-\frac{\tau}{\mu}} I\right)$

$$\Rightarrow \frac{d}{d\tau}\left(e^{-\frac{\tau}{\mu}} I\right) = -e^{-\frac{\tau}{\mu}} \frac{S}{\mu}$$

$$e^{-\frac{\tau}{\mu}} I \Big|_{\tau_0}^{\tau} = -\int_{\tau_0}^{\tau} e^{-\frac{t}{\mu}} \frac{S(t)}{\mu}\, dt$$

τ_0 : reference point, $\tau_0 = 0$: surface

$\tau_0 = \infty$: center of star

t: dummy integration variable

$$e^{-\frac{\tau}{\mu}} I(\tau) - e^{-\frac{\tau_0}{\mu}} I(\tau_0) = -\int_{\tau_0}^{\tau} e^{-\frac{t}{\mu}} \frac{S(t)}{\mu}\, dt$$

multiply $e^{\tau/\mu}$ on both sides

$$\Rightarrow I(\tau, \mu) = e^{-(\tau_0 - \tau)/\mu} I(\tau_0, \mu) + \int_{\tau}^{\tau_0} e^{-\frac{(t-\tau)}{\mu}} \frac{S(t)}{\mu}\, dt \quad \text{--- (4.11)}$$

— the general solution

* Consider a point τ at depth of the star

* Forward Directed Radiation $I(\tau, \mu \geq 0)$; $0 \leq \theta \leq \frac{\pi}{2}$

one can choose the reference point $\tau_0 = \infty$, the center

$$e^{-(\tau_0 - \tau)/\mu} = e^{-\infty} = 0$$

$$I(\tau, \mu \geq 0) = \int_{\tau}^{\infty} e^{-(t-\tau)/\mu} \frac{S(t)}{\mu}\, dt \quad \text{--- (4.12)}$$

* Inward directed radiation $I(\tau, \mu \le 0)$
$\frac{\pi}{2} \le \theta \le \pi$, $\mu = \cos\theta$

The reference point $\tau_0 = 0$. the surface.

At $\tau_0 = 0$, $I(0, \mu \le 0) = 0$
(The inward radiation at the surface $= 0$)

$$\Rightarrow I(\tau, \mu \le 0) = \int_\tau^0 e^{-(t-\tau)/\mu} \frac{S(t)}{\mu} dt \quad \text{--- (4.13)}$$

* To solve (4.12) and (4.13), need to specify $S(t)$.
Assuming the source function $S(t)$ at t is near the one $S(\tau)$ at τ.
A reasonable assumption, the source t not far from τ
Assuming $S(\tau) = B(\tau)$: LTE + full isotropic
* Use Taylor expansion, and keep the first order

$$S(t) = B(\tau) + (t-\tau)\left(\frac{\partial B}{\partial t}\right)_\tau$$

(4.12) becomes

$$I(\tau, \mu \ge 0) = \int_\tau^\infty e^{-(t-\tau)/\mu}\left(\frac{B(\tau)}{\mu} + \frac{(t-\tau)}{\mu}\left(\frac{\partial B}{\partial t}\right)_\tau\right) dt$$

The first term : $\frac{B(\tau)}{\mu}\int_\tau^\infty e^{-(t-\tau)/\mu} dt = -\frac{B(\tau)\mu}{\mu} e^{-\frac{t-\tau}{\mu}}\Big|_\tau^\infty$

$$= -B(\tau)(0-1) = B(\tau)$$

The second term $\left(\frac{\partial B}{\partial t}\right)_\tau \int_\tau^\infty e^{-(t-\tau)/\mu}\left(\frac{t-\tau}{\mu}\right) dt$

$x = \frac{t-\tau}{\mu}$ $\quad = \mu\left(\frac{\partial B}{\partial t}\right)_\tau \int_0^\infty e^{-x} x\, dx$

$$\int_0^\infty e^{-x} x\, dx = -\int_0^\infty x\, de^{-x} = -\left[x e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x} dx\right]$$

$$= \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$$

$$\Rightarrow I(\tau, \mu \ge 0) = B(\tau) + \mu\left(\frac{\partial B}{\partial t}\right)_\tau \quad > B(\tau) \quad \text{--- (4.15)}$$

* (4.13), similar process

$$I(\tau, \mu \le 0) = \int_\tau^0 e^{-\frac{t-\tau}{\mu}} \left(\frac{B(\tau)}{\mu} + \frac{(t-\tau)}{\mu} \left(\frac{\partial B}{\partial \tau}\right)_\tau \right) dt$$

First term: $B(\tau) \int_\tau^0 e^{-\frac{t-\tau}{\mu}} \frac{dt}{\mu} = -B(\tau) e^{-\frac{t-\tau}{\mu}} \Big|_\tau^0$

$$= -(B(\tau)[e^{\tau/\mu} - 1] = B(\tau)(1 - e^{\frac{\tau}{\mu}})$$

Second term $\left(\frac{\partial B}{\partial \tau}\right)_\tau \int_\tau^0 e^{-\frac{t-\tau}{\mu}} \frac{t-\tau}{\mu} dt = \left(\frac{\partial B}{\partial \tau}\right)_\tau \mu \int_0^{-\frac{\tau}{\mu}} e^{-x} x\, dx$

$$\int_0^{-\frac{\tau}{\mu}} e^{-x} x\, dx = -\int x\, de^{-x} = -\left[x e^{-x} \Big|_0^{-\frac{\tau}{\mu}} - \int e^{-x} dx \right]$$

$$= \frac{\tau}{\mu} e^{+\frac{\tau}{\mu}} - e^{-x} \Big|_0^{-\frac{\tau}{\mu}} = e^{\frac{\tau}{\mu}} \left(\frac{\tau}{\mu} - 1\right) + 1$$

$$\Rightarrow I(\tau, \mu \le 0) = B(\tau)[1 - e^{\tau/\mu}] + \mu \left(\frac{\partial B}{\partial \tau}\right)_\tau \left[e^{\frac{\tau}{\mu}} \left(\frac{\tau}{\mu} - 1\right) + 1 \right] \quad \text{---} (4.16)$$

Since we are consider at large depth $\tau \gg 0$

and $\mu \le 0 \Rightarrow \tau/\mu \ll 0 \Rightarrow e^{\frac{\tau}{\mu}} = 0$

$$\Rightarrow I(\tau, \mu \le 0) = B(\tau) + \mu \left(\frac{\partial B}{\partial \tau}\right)_\tau < B(\tau)$$

the same as $I(\tau, \mu \ge 0)$

Radiation Flux in the interior of a star

$$F(\tau) = 2\pi \int_{-1}^{+1} I(\mu) \mu\, d\mu$$

$$= 2\pi \int_{-1}^{+1} \Big(\underbrace{\mu B(\tau)}_{\frac{1}{2}\mu^2 \big|_{-1}^{+1} = 0} + \underbrace{\mu^2 \left(\frac{\partial B}{\partial \tau}\right)_\tau}_{\frac{1}{3}\mu^3 \big|_{-1}^{+1} = \frac{2}{3}} \Big) d\mu$$

$$\Rightarrow \boxed{F(\tau) = \frac{4\pi}{3} \frac{\partial B(\tau)}{\partial \tau}} \quad \text{------} \quad (4.17)$$

net radiation flux

$L = 4\pi r^2 F(r)$ --- luminosity --- 4.18

$\boxed{B(r) \equiv \frac{\sigma T(r)^4}{\pi}}$ integrated Planck function

* Deviation from Isotropy of $I(\theta)$

$$F_{net} = \frac{4\pi}{3}\frac{\partial B(\tau)}{\partial \tau}$$

$$F_{out} = F(\mu \geq 0) = 2\pi \int_0^1 I(\mu)\, \mu\, d\mu$$

$$= 2\pi \int_0^1 \left(\mu B(\tau) + \mu^2 \left(\frac{\partial B}{\partial \tau}\right)_\tau\right) d\mu$$

$$\frac{1}{2}\mu^2\Big|_0^1 = \frac{1}{2} \qquad \frac{1}{3}\mu^3\Big|_0^1 = \frac{1}{3}$$

$$= \pi B(\tau) + \frac{2\pi}{3}\left(\frac{\partial B}{\partial \tau}\right)_\tau$$

Similarly $F_{in} = F(\mu \leq 0) = 2\pi \int_{-1}^0 \left(\mu B(\tau) + \mu^2\left(\frac{\partial B}{\partial \tau}\right)_\tau\right) d\mu$

$$\frac{1}{2}\mu^2\Big|_{-1}^0 = -\frac{1}{2} \qquad \frac{1}{3}\mu^3 = \frac{1}{3}$$

$$= -\pi B(\tau) + \frac{2\pi}{3}\left(\frac{\partial B}{\partial \tau}\right)_\tau$$

Total net flux $F_{net} = F_{out} + F_{in} = \frac{4\pi}{3}\left(\frac{\partial B}{\partial \tau}\right)_\tau$

Total absolute flux $F_{abs} = |F_{out}| + |F_{in}| = 2\pi B(\tau)$

Anisotropy $\delta = \frac{F_{net}}{F_{abs}} = \frac{\frac{4\pi}{3}\frac{\partial B}{\partial \tau}}{2\pi B(\tau)} = \frac{2}{3}\frac{\frac{\partial B}{\partial \tau}}{B(\tau)}$

For the Sun: $F_{net} = \frac{L_\odot}{4\pi R^2} = \frac{4 \times 10^{33}}{4\pi (6.96 \times 10^{10})^2} = 2 \times 10^{10}\ \mathrm{erg\, cm^{-2}\, s^{-1}}$

$F_{abs} = 2\pi \cdot \frac{\sigma T_\odot^4}{\pi}$ $\qquad T_\odot = 10^7\ \mathrm{K}$

$\sigma = 5.67 \times 10^{-5}\ \mathrm{erg\, cm^{-2}\, K^{-4}\, s^{-1}}$

$\approx 10^{23}\ \mathrm{erg\, cm^{-2}\, s^{-1}}$

$$\delta \approx \frac{10^{10}}{10^{23}} = 10^{-13}.$$

Thus, a small deviation from isotropic LTE.
This deviation is caused by temperature gradient

$\frac{\partial B}{\partial \tau} \propto \frac{dT}{dr}$; $\quad B = \sigma T^4/\pi$

$d\tau = -k\rho\, dr$

Eddington Critical Luminosity or Eddington Limit
(End of CH 4.3)

$$F = \frac{4\pi}{3}\frac{\partial B}{\partial \tau} = -\frac{4ac}{3}\frac{1}{\kappa\rho}T^3\frac{dT}{dr}$$

$$\Rightarrow F = -\frac{c}{\kappa\rho}\frac{d(\frac{1}{3}aT^4)}{dr} = -\frac{c}{\kappa\rho}\frac{dP_{rad}}{dr}$$

Since Radiation Pressure $P_{rad} = \frac{1}{3}aT^4$

$$\Rightarrow L = 4\pi R^2 F = -\frac{4\pi R^2 c}{\kappa\rho}\frac{dP_{rad}}{dr} \quad \text{--- (4.51)}$$

Eddington limit: radiation pressure force exceeds the gravitational force, resulting in non-hydrodynamic equilibrium

$$\boxed{-\frac{dP_{rad}}{dr} = \rho g_s} \quad \text{--- 4.50}$$

$$L_{Edd} = \frac{4\pi R^2 c}{\kappa\rho}\,\rho g_s \quad \frac{GM^2}{R^2}$$

$$\boxed{L_{Edd} = \frac{4\pi c GM}{\kappa}} \quad \text{--- (4.52)}$$

Use $\kappa = \kappa_e = 0.2(1+x) = 0.34 \ cm^2 g^{-1}$
opacity due to Thomson electron scattering
and $x = 0.7$

$$\frac{L_{Edd}}{L_\odot} = 3.5 \times 10^4 \left(\frac{M}{M_\odot}\right) \quad \text{--- 4.53)}$$

* For current Sun, Eddington effect is small
* However, for supergiants, radiation pressure drives strong stellar wind → mass loss → planetary nebula

4.2 The Diffusion Equation (radiation)

$$F_\nu = \frac{4\pi}{3}\frac{\partial B_\nu}{\partial \tau} \quad \text{------} \quad (4.19)$$

→ the diffusion form : (Fick's Law of Diffusion)

$$F_\nu = -D\frac{d\phi}{dr} \quad \text{or} \quad F_\nu = -D_T\frac{dT}{dr}$$

$$\frac{\partial B}{\partial \tau} = \frac{\partial B}{\partial T}\frac{\partial T}{\partial \tau}$$

$$B = \frac{\sigma T^4}{\pi} = \frac{c}{4\pi}aT^4 \Rightarrow \frac{\partial B}{\partial T} = \frac{ca}{\pi}T^3$$

$$\partial\tau = -\kappa\rho\,\partial r \Rightarrow \frac{\partial T}{\partial \tau} = -\frac{1}{\kappa\rho}\frac{\partial T}{\partial r}$$

$$\Rightarrow F(r) = -\frac{4ac}{3}\frac{1}{\kappa\rho}T^3\frac{dT}{dr} \quad \text{---} \quad (4.2]$$

$$\Rightarrow F(r) = -\frac{c}{3\kappa\rho}\cdot\frac{d(aT^4)}{dr} \quad \text{----} \quad (4.23)$$

The diffusion equation

$\Phi = aT^4$; the concentration is radiation energy

$$D = \frac{c}{3\kappa\rho} = \frac{1}{3}c\lambda_{free}$$

$$\Rightarrow \boxed{F(r) = -\frac{1}{3}c\lambda\frac{du}{dr}}$$

Luminosity

$$L(r) = 4\pi r^2 F(r) = -\frac{16\pi acr^2}{3\kappa\rho}T^3\frac{dT}{dr} \quad \text{--} \quad (4.24)$$

4.5 Heat Transfer by conduction

— Diffusion equation of Heat

F_{cond} : conduction flux due to electron's random kinetic energy

$$F_{cond} = -\frac{1}{3} V_e \lambda_e \frac{dQ}{dT}$$ -- similar to radiation

$Q = C_v T$, the concentration

$$\begin{cases} C_v = \frac{3}{2}\frac{N_A k}{\mu} & \text{for ideal gas} \\ C_v = \frac{8\pi^3 m_e^2 c}{3h^3} k^2 T x_f & \text{---- (4.71)} \end{cases}$$

V_e : electron velocity

λ_e : electron collisional mean free path

$$\lambda_e = \frac{1}{\sigma_c n_I}$$

n_I : ion number density

σ_c : Coulomb scattering cross section

$\sigma_c = \pi s^2$. s : impact parameter

$m_e V_e^2 = \frac{Z e^2}{s}$: significant scattering as kinetic energy is about the same as the electrostatic potential energy

$$\Rightarrow F_{cond} = -D_e \frac{dT}{dr}$$

$$F_{total} = F_{rad} + F_{cond}$$

* Conduction flux is negligible in main sequence stars, but important in white dwarfs (high density)

CH4.4 Radiative Opacity Sources

κ : determined by microphysics

Unit : $cm^2 g^{-1}$

$$d\tau = -\kappa\rho\, dr \quad ; \quad \kappa\rho = \frac{1}{\lambda_{free}}$$

For star, high κ, $\Rightarrow$ (1) high temperature gradient

(2) low luminosity

* $\kappa_{total} = \kappa_e + \kappa_{ff} + \kappa_{bf} + \kappa_{H^-}$

κ_e : electron scattering — high T

κ_{ff} : free - free absorption

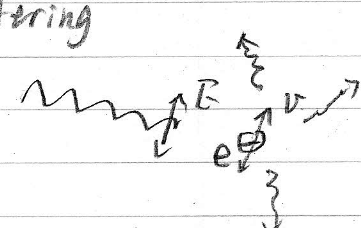
κ_{bf} : bound - free absorption

κ_{H^-} : metal hydrogen absorption — low T

Electron Scattering κ_e

In classical EM, scattering is caused by oscillation of free electrons upon the incidence of electro-magnetic radiation,

* — so called "Thomson Scattering"



scattering cross section :

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = \frac{8\pi}{3} r_e^2$$

$$= 0.6652 \times 10^{-24}\ cm^2 \quad \text{--- (4.59)}$$

$r_e = \frac{e^2}{m_e c^2}$: classical electron radius

Cross section quantifies the probability of absorption for a given particle:

probability of absorption per unit volume : $\sigma_e n_e$. cm^{-1}

Mean free path : $\lambda_{free}\sigma_e n_e = 1$ (probability = 1)

$$\lambda_{free} = \frac{1}{\sigma_e n_e} = \frac{1}{\kappa \rho}$$

$$\Rightarrow \kappa_e = \frac{\sigma_e n_e}{\rho}$$

Since $n_e = \frac{\rho}{\mu_e} N_A$

$$\Rightarrow \kappa_e = \frac{\sigma_e N_A}{\mu_e}$$

Consider only H and all other elements

$$\mu_e = \left(\Sigma \frac{X_i Y_i Z_i}{A_i}\right)^{-1} = \left(\frac{X \cdot 1 \cdot 1}{1} + \frac{(1-X) \cdot 1 \cdot \cancel{Z}}{2\cancel{Z}}\right)^{-1}$$

$$\mu_e = \frac{2}{1+X}$$

$$\Rightarrow \kappa_e = 0.2(1+X) \quad cm^2 g^{-1} \quad \cdots \quad (4.62)$$

$\kappa_e = 0.34 \ cm^2 g^{-1}$ for a ZAMS $(X = 0.7)$

Exp: Solar core: $X = 0.7$, $\rho = 150 \ g \ cm^{-3}$

$$\kappa \rho = 0.34 \times 150 = 51 \ cm^{-1}$$

$\lambda_{free} = \frac{1}{\kappa \rho} = 0.02 \ cm$: photon free path

$\Rightarrow$ * Thomson scattering is the most common opacity in stellar interior where matter is fully ionized.

* In power-law notation

$$\kappa_e = \kappa_0 \rho^n T^{-s}$$

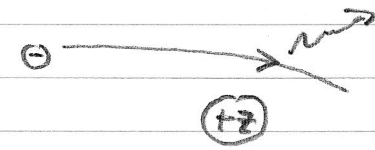
$$n = 0, \ s = 0$$

* κ_{ff}: free-free absorption

Inverse process of free-free emission,

Bremsstrahlung emission

Breaking emission



* Free-Free emission: produced by electron radiation when an electron accelerates due to electrostatic Coulomb force; the electron-ion interaction

* Free-free absorption: photons are absorbed by free electrons

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}a^2(t) \quad \text{(Larmor result of radiation power)}$$

$$4\pi j \rho = \frac{2\pi}{3}\frac{Z_c^2 e^6}{m_e c^3 \hbar}\left(\frac{2\pi kT}{m_e}\right)^{\frac{1}{2}} n_e n_I$$

$$\kappa_{ff} = 10^{23}\frac{\rho}{\mu_e}\frac{Z_c^2}{\mu_I}T^{-3.5} \quad cm^2 g^{-1} \quad \text{— (4.61)}$$

In power-law notation (Kramer's Opacity):

$$\boxed{\begin{array}{c}\kappa_{ff} = \kappa_0 \rho T^{-3.5} \\ n=1, \; s=3.5\end{array}}$$

κ_{ff}: high in stellar interior, but prefer low T.