Oct. 16, 2012

3.5.2 . Application to White Dwarfs

known to be supported by degenerate electron pressure

non-relativistic limit

$$E_e = A g(x) \sim x^5 \qquad x \ll 1$$

$$\frac{\rho}{\mu_e} = B x^3 \sim x^3$$

$$E_e \propto \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}$$

Total internal energy $U = E_e V \propto \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}} R^3$

$$\rho \propto M R^{-3}$$

$$U \propto \frac{M^{\frac{5}{3}}}{\mu_e^{5/3}} \frac{1}{R^2}$$

Virial theorem $3(\gamma-1)U = -\Omega = -q\frac{GM^2}{R}$

$$\Rightarrow M \propto \frac{1}{R^3} \frac{1}{\mu_e^5}$$

The exact mass-radius relation

$$M = \frac{1}{4}\left(\frac{3}{4\pi}\right)^4 \left(\frac{h^2 N_A}{m_e G}\right)^3 \frac{N_A^2}{\mu_e^5} \frac{1}{R^3} \quad \text{-- (3.62)}$$

$$M \propto \frac{1}{R^3}, \text{ or } R \propto M^{-\frac{1}{3}}$$

much different from main sequence stars, $R \propto M^{0.75}$

Difference is due to different equation of state.

* plug-in constants $\frac{M}{M_\odot} = 10^{-6}\left(\frac{R}{R_\odot}\right)^{-3}\left(\frac{2}{\mu_e}\right)^5$ --(3.63)

Complete ionization of He, C, O, $\mu_e = 2$

$M = M_\odot$

$\Rightarrow \boxed{R \simeq 0.01\, R_\odot}$ size of a typical white dwarf

$\Rightarrow R = 7000$ km Earth-size

* Similarly, for neutron star

$$\frac{M}{M_\odot} = 5 \times 10^{-15} \left(\frac{R}{R_\odot}\right)^{-3} \qquad \text{--- (3.64)}$$

$M = M_\odot \quad \Rightarrow \boxed{R = 11 \text{ km}}$

city size

* Chandrasekhar limit M_∞

$M_\infty = 1.456\, M_\odot$ From analytic calculation

$M \propto R^{-3}$, $M \uparrow \Rightarrow R \downarrow$

what $R \to 0$, or negative

R – M relationship including the relativistic effect

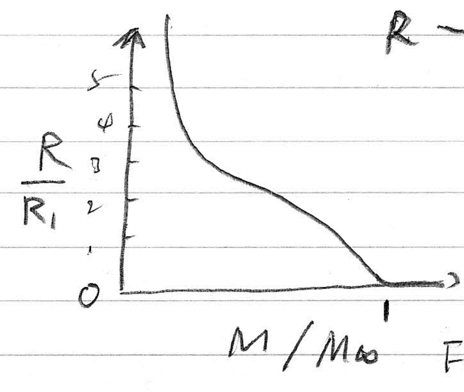

Figure 3.6

$$\frac{M_\infty}{M_\odot} = 1.456 \left(\frac{2}{\mu_e}\right)^2 \qquad \text{--- (3.67)}$$

$$\frac{R_1}{R_\odot} = 2.02 \left[1 - \left(\frac{M}{M_\infty}\right)^{\frac{4}{3}}\right]^{\frac{1}{2}} \left(\frac{M}{M_\infty}\right)^{-\frac{1}{3}} \qquad (3.68)$$

3.5.3 Effect of Temperature on Degeneracy

$$n_e = \frac{8\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} x_F^{\,3} \quad \cdots (3.44)$$

$$x_F = \frac{P_F}{mc}$$

$$\varepsilon_F = \mu - mc^2 = \varepsilon(p_F) = mc^2\left[1+\left(\frac{P_F}{mc}\right)^2\right]^{\frac{1}{2}} - 1$$

Density $\uparrow \Rightarrow x_F \uparrow \Rightarrow \varepsilon_F \uparrow$

high density, high x_F, $\frac{kT}{\varepsilon_F} \ll 1 \Rightarrow T \sim 0$, completely degenerate

low density, low x_F, $\frac{kT}{\varepsilon_F} \gg 1 \Rightarrow T \to \infty$, complete thermal

$\frac{kT}{\varepsilon_F} \sim 1$: mixture of degeneracy pressure and thermal pressure

* the boundary : $\varepsilon_F = kT \Rightarrow \rho_{critical}$

non-relativistic

$$\varepsilon_F = mc^2\left[(1+x_F^2)^{\frac{1}{2}} - 1\right] \quad -- (3.48)$$

$$\Rightarrow \varepsilon_F = mc^2 \frac{x_F^2}{2} = kT$$

$$\frac{\rho}{\mu_e} = 9.739 \times 10^5 x_F^{\,2} \quad \text{------} (3.52)$$

$$\Rightarrow \frac{\rho_{crit}}{\mu_e} = 6.0 \times 10^{-9}\ T^{3/2}\ \text{g cm}^{-3} \quad -- (3.70)$$

White Dwarf : $T = 10^8$ K, $\rho_{crit} = 6.0 \times 10^3$ g cm^{-3} → Dege

On the Earth $T = 300$ K $\rho_{crit} = 10^{-4}$ g cm^{-3}

metals : free electrons due to pressure ionization

$\rho = 5$ g cm^{-3} $\gg \rho_{crit}$ → Degerate

3.7. Adiabatic Exponents and Other Derivatives.

Given state parameters: $\underline{T}$, $\underline{n.\rho}$, $\underline{p.E}$,

very often $p = p(\rho, T)$

$E = E(\rho, T)$.

* Exists the derivative of these parameters, which characterize the thermodynamic system

* Specific heats

$$C_v = \left(\frac{dQ}{dT}\right)_v = \left(\frac{dQ}{dT}\right)_\rho$$

constant volume $= V_\rho$: volume per unit mass ($cm^3\ g^{-1}$), specific volume

$$V_\rho = \frac{1}{\rho}$$

$\Rightarrow$ constant mass density

$$C_p = \left(\frac{dQ}{dT}\right)_p$$

* Ratio of specific heats : γ index (dimensionless)

$$\gamma = \frac{C_p}{C_v}$$

$$dQ = dE + p\,dV = dE + p\,d\left(\frac{1}{\rho}\right)$$

$$dQ = dE - \frac{p}{\rho^2}\,d\rho \qquad (3.84)$$

From state equations, one know $E, p, \rho \Rightarrow Q$

$\Rightarrow C_v$, C_p and γ for different systems

$$C_v = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$$

For ideal gas : $E = \frac{3}{2} n k T = \frac{3}{2} \frac{\rho N_A}{\mu} k T$

$$C_V = \frac{3}{2} \frac{N_A k}{\mu} \rho \quad \text{(unit volume)}$$

$$C_V = \frac{3}{2} \frac{N_A k}{\mu} \quad \text{(unit mass)}$$

* From thermodynamics text

$$C_P - C_V = -T \left(\frac{\partial P}{\partial T}\right)_V^2 \left(\frac{\partial P}{\partial V}\right)_T^{-1} \quad \text{--- (3.85)}$$

* Power Law expression of pressure.
general expression for pressure (ideal, radiation)

$$P = P_0 \rho^{\chi_\rho} T^{\chi_T} \quad \text{--- (3.88)}$$

For ideal gas : $\chi_\rho = 1$, $\chi_T = 1$

For radiation : $\chi_\rho = 0$, $\chi_T = 4$

For degenerate $\begin{cases} \chi_\rho = \frac{5}{3}, \ \chi_T = 0 & x \ll 1 \\ \chi_\rho = \frac{4}{3}, \ \chi_T = 0 & x \gg 1 \end{cases}$

Logarithm of (3.88)

$$\ln P = \ln P_0 + \chi_\rho \ln \rho + \chi_T \ln T$$

Differentiation $d\ln P = \chi_\rho \, d\ln \rho + \chi_T \, d\ln T$

$$\Rightarrow \chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_T$$

$$\Rightarrow \chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_\rho$$

plug-in (3.85) $C_P - C_V = \frac{P}{\rho T} \frac{\chi_T^2}{\chi_\rho}$ erg g^{-1} K^{-1} (3.92)

For ideal gas $\chi_\rho = 1$, $\chi_T = 1$

$$C_P - C_V = \frac{N_A k}{\mu} \quad \text{erg g}^{-1}\text{ K}^{-1}$$

Adiabatic Exponents "Γs"

when the system undergoes adiabatic change, the state parameters are constrained by the adiabatic process : $dQ = 0$, when ρ changes

e.g. convection motion

sound waves

☆ One could introduce the dimensionless thermodynamic derivatives Γ_1, Γ_2, Γ_3

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{ad}$$: how pressure changes with ρ

$$\Rightarrow \ln P = c + \Gamma_1 \ln \rho$$

$$\Rightarrow P = c\rho^{\Gamma_1}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad}$$ P versus T

$$\Rightarrow P = c\,T^{\frac{\Gamma_2}{\Gamma_2 - 1}}$$

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{ad}$$ T versus ρ

$$T = c\rho^{\Gamma_3 - 1}$$

$$\frac{\Gamma_3 - 1}{\Gamma_1} = \frac{\left(\frac{\partial \ln T}{\partial \ln \rho}\right)}{\left(\frac{\partial \ln P}{\partial \ln \rho}\right)} = \left(\frac{\partial \ln T}{\partial \ln P}\right) = \frac{\Gamma_2 - 1}{\Gamma_2} = \nabla_{ad}$$

∇_{ad} : adiabatic temperature gradient normalized to pressure in LOG scale

These exponents can be obtained by the derivatives of state parameters:

$$\Gamma_3 - 1 = \frac{p}{\rho T}\frac{\chi_T}{C_v} = \frac{1}{\rho}\left(\frac{\partial p}{\partial E}\right)_\rho \quad \text{--- (3.97)}$$

$$\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\rho \quad \text{--- (3.98)}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = C_p \frac{\rho T}{p}\frac{\chi_\rho}{\chi_T} = \frac{\chi_\rho}{\Gamma_3 - 1} + \chi_T \quad \text{--- (3.99)}$$

$$\gamma = \frac{C_p}{C_v} = \frac{\Gamma_1}{\chi_\rho} = 1 + \frac{\chi_T}{\chi_\rho}(\Gamma_3 - 1) \quad \text{--- (3.100)}$$

* From (3.97) $p = (\Gamma_3 - 1)\rho E$,

So exactly, the γ-law equation of state concerns Γ_3

$$\Gamma_3 - 1 = \frac{p}{\rho T}\frac{\chi_T}{C_v}$$

* Ideal gas $\quad C_v = \frac{3}{2}\frac{N_A k}{\mu} \qquad \chi_\rho = 1, \chi_T = 1$

$$C_p = \frac{5}{2}\frac{N_A k}{\mu}$$

$$\gamma = \frac{C_p}{C_v} = \frac{5}{3} : \text{ ratio of specific heats}$$

$$\Gamma_3 - 1 = \frac{2}{3}, \quad \Gamma_3 = \frac{5}{3}$$

$$\Gamma_1 = \Gamma_3 = \frac{5}{3}. \quad \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{1}{\frac{5}{3} - 1} + 1 = \frac{3}{2} + 1 = \frac{5}{2}$$

$$\Gamma_2 = \frac{5}{3}$$

$$\Rightarrow \gamma = \Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{5}{3} \text{ interchangable}$$

* Radiation : $\chi_\rho = 0, \quad \chi_T = 4$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = 4/3$$

$$\gamma = \Gamma_1/\chi_\rho = \infty$$

Mixture of Ideal gas and Radiation

What are χ_ρ, χ_T ? $\qquad P = P_0 \rho^{\chi_\rho} T^{\chi_T}$

What are Γ_1, Γ_2, Γ_3, γ ?

$$P = P_g + P_{rad} \qquad \text{dyne cm}^{-2}$$

$$P = \frac{\rho N_A}{\mu} kT + \frac{aT^4}{3} \qquad \text{--- (3.104)}$$

$$E = \frac{3}{2}\frac{N_A kT}{2\mu} + \frac{aT^4}{\rho} \quad \text{erg g}^{-1} \qquad \text{--- (3.105)}$$

* Introduce β number : $\beta \equiv \dfrac{P_g}{P}$

$\beta = 1$. gas pressure dominates

Looking for $d\ln P = \chi_\rho \, d\ln\rho + \chi_T \, d\ln T$

or $\dfrac{dP}{P} = \chi_\rho \dfrac{d\rho}{\rho} + \chi_T \dfrac{dT}{T}$

Differentiate (3.104): $dP = dP_g + dP_{rad}$

$$\frac{dP}{P} = \frac{dP_g}{dP} + \frac{dP_{rad}}{dP} = \beta \frac{dP_g}{P_g} + (1-\beta)\frac{dP_{rad}}{dP_{rad}}$$

For ideal gas $\dfrac{dP_g}{P_g} = 1\dfrac{d\rho}{\rho} + 1\dfrac{dT}{T}$

For radiation $\dfrac{dP_{rad}}{P_{rad}} = d\ln P_{rad} = 4\dfrac{dT}{T} = 4\, d\ln T$

$$\Rightarrow \frac{dP}{P} = \beta\frac{d\rho}{\rho} + (4-3\beta)\frac{dT}{T}$$

$$\Rightarrow \chi_\rho = \beta \qquad \text{--- (3.106)}$$

$$\chi_T = 4 - 3\beta \qquad \text{--- (3.107)}$$

Further $\Gamma_3 - 1 = \dfrac{2}{3}\left(\dfrac{4-3\beta}{8-7\beta}\right)$

$$\Gamma_1 = \beta + (4-3\beta)(\Gamma_3 - 1), \quad \gamma = \Gamma_1/\beta \qquad \text{--- (3.112)}$$