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CH3. Equations of state

* State : statistical properties of individual particles within the system.

* State parameters : T, P, n, ρ, E, S

* State equations for photons (or radiation (CH3.2)
for ideal gas particles (CH3.3)
for ionization (Saha Eq.) (CH3.4)
for degenerate particles (CH3.5)
(non-ideal : the quantum effect)

* Adiabatic Exponents :
constants that relate state parameters

e.g. $\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{ad}$

$\gamma = \frac{C_p}{C_v}$

* Distribution function : the control function that derives the state parameters

In general : $f(\vec{x}, \vec{v}, t) = f(x, y, z, v_x, v_y, v_z, t)$

uniform distribution : $\vec{x}$ dependent is gone

isotropic distribution : $(v_x, v_y, v_z) \longrightarrow p$ (momentum)

$$\Rightarrow n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\{[-\mu + \varepsilon_j + \varepsilon(p)]/kT\} \pm 1} \qquad (3.9)$$

T : temperature, a parameter that is valid only satisfying the "LTE" assumption

LTE : local thermal equilibrium

LTE (Local Thermal Equilibrium) (CH3 : Introduction)

→ in order to have "T" well defined

In order to achieve the equilibrium, particles should experience sufficient interaction with each other

① partice interaction time scale << characteristic time scale of the system

e.g. $t_{collision} << t_{dyn}$

② particle mean free path << characteristic length scale of the system

e.g. $\lambda_{free} << \lambda_p$

λ_p: pressure scale height

* λ_{free}: photon mean free path

$$\lambda_{free} = (\kappa\rho)^{-1} \text{ cm} \quad \text{------} \quad (3.2)$$

κ: opacity ($cm^2\, g^{-1}$)

For opacity of free electrons (smallest opacity)
— the Thomson electron scattering.

For single electron $\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.6652 \times 10^{-24}\ cm^2$

In a star, e.g. $\langle\rho_\odot\rangle = 1.4\ g\ cm^{-3}$

$\mu_e = 1.2$

$$\kappa = \underbrace{\frac{1(g)}{\mu_e} N_A}_{\#\text{ electrons per gram}} \cdot \sigma_e = \frac{1}{1.2} \cdot 6\times10^{+23} \cdot 0.6\times10^{-24} = \underbrace{0.3\ cm^2 g^{-1}}_{\text{opacity per gram}}$$

$\kappa\rho \sim 1\ cm^{-1}$

$\Rightarrow \lambda_{free} = 1\text{ cm}$

$\lambda_{free} << \lambda_p \Rightarrow$ valid LTE assumption

* λ_p: pressure scale height

$$P = P_0 e^{-\frac{r}{\lambda_p}} \Rightarrow \lambda_p : \text{e-folding length of pressure}$$

$$\ln P = \ln P_0 - \frac{r}{\lambda_p}$$

$$d\ln P = -\frac{1}{\lambda_p} dr \quad \text{since } d\ln P_0 = 0$$

$$\lambda_p = -\left(\frac{d\ln P}{dr}\right)^{-1}$$

$$d\ln P = \frac{1}{P} dP \qquad \frac{dP}{dr} = -\rho g$$

$$\Rightarrow \lambda_p = -\left(\frac{1}{P}\cdot(-\rho g)\right)^{-1}$$

$$\Rightarrow \lambda_p = \frac{P}{\rho g} \qquad \text{———— (3.1)}$$

For stars, for constant density model.

$$\lambda_p = \frac{R^2}{2r}\left[1 - \frac{r}{R}\right]^2$$

At center, $r=0$, $\lambda_p \to \infty$

At surface $r=R$ $\lambda_p \to 0$

At most r, $\lambda_p \approx R$

Thus $\lambda_p \gg \lambda_{free}$.

Constant density model and λ_p.

$$P = P_c\left[1 - \frac{r}{R}\right]^2$$

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

$$\rho = \rho_{const} = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$g = \frac{GM_r}{r^2} = \frac{G\cdot\frac{4}{3}\pi r^3 \rho}{r^2} = \frac{4\pi}{3}G\rho r$$

CH3.1 Distribution function

Also called phase space density

$$f(x, y, z, v_x, v_y, v_z, t) = f(\vec{x}, \vec{v}, t)$$

for a uniform, isotropic medium

$$f(\vec{x}, \vec{v}, t) \longrightarrow n(p) \text{ (density in momentum space)}$$

From statistical mechanics:

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\{(-\mu + \varepsilon_j + \varepsilon(p))/kT\} \pm 1} \quad \text{--- (3.9)}$$

Two constants: $h = 6.626 \times 10^{-27}$ erg s (Planck const)

$k = 1.38 \times 10^{-16}$ erg K^{-1} (Boltzmann const)

$\varepsilon(p)$: kinetic energy of particles

p: momentum of particles

ε_j: energy of excitation state j

(refer to a reference level. e.g. H. $j=1, \varepsilon_1 = -13.6$ eV; $j=2, \varepsilon_2 = -3.4$ eV; $j=3, \varepsilon_3 = -1.5$ eV)

μ: chemical potential of the species

necessary when chemical reaction occurs

One species to another, energy changes

e.g. $H^+ + e^- \rightleftharpoons H^0 + \gamma$

$$\mu_{H^+} + \mu_e = \mu_H + \chi_H$$

χ_H: ionization energy

i: number of energy states in different excitation

g_i: number of degeneracy of energy state i $(2n^2)$

+ : Fermi-Dirac particles — half integer spin $s=\frac{1}{2}$
e.g. p, e, n
$n(p)$: called Fermi-Dirac distribution

− : Boson-Einstein particles
— zero or integer spin $s=0, 1$
e.g. photons γ
$n(p)$: called Boson-Einstein distribution

Derive state parameters from $n(p)$

$$n = \int_p n(p)\, 4\pi p^2\, dp \quad cm^{-3} \quad \text{--- } (3.10)$$

$4\pi p^2 dp$: momentum shell
$\int_0^\infty \cdots 4\pi p^2 dp$: integrate over momentum sphere

$$P = \int_p n(p) \left(\tfrac{1}{3} p v\right) 4\pi p^2\, dp \quad \text{--- } (3.13)$$

momentum flux cross the surface
$\frac{1}{3}$: integration coeffient in all directions

$$E = \int_p n(p) \left(\varepsilon(p)\right) 4\pi p^2\, dp \quad \text{--- } (3.14)$$

For ideal particles: $p = mv$
$$\varepsilon(p) = \frac{1}{2} m v^2 = \frac{1}{2} p v$$

For all particles, including Fermions and Bosons ($m=0$)
non-ideal such as relativistic

$$\varepsilon(p) = (p^2 c^2 + m^2 c^4)^{\frac{1}{2}} - mc^2 \quad \text{--- } (3.11)$$

$$v = \frac{\partial \varepsilon}{\partial p} \quad \text{--- } (3.12)$$

m : rest mass

* non-relavistic gas particles ($m \neq 0$)

$v << c, \quad m \neq 0$

$$\varepsilon(p) = mc^2 \left[1 + \left(\frac{p}{mc}\right)^2 \right]^{\frac{1}{2}} - mc^2$$

since $\frac{p}{mc} << 1 \Rightarrow \varepsilon(p) = \frac{1}{2}\left(\frac{p}{mc}\right)^2 \cdot mc^2$

$$\varepsilon(p) = \frac{1}{2}\frac{p^2}{m}$$

$$v = \frac{\partial \varepsilon}{\partial p} = \frac{p}{m}$$

$$\Rightarrow \boxed{p = mv}$$

$$\Rightarrow \boxed{\varepsilon = \frac{1}{2} m v^2}$$

photons $v = c \quad m = 0$

$$\boxed{\varepsilon(p) = pc}$$

$$v = \frac{\partial \varepsilon}{\partial p} = c$$

If $\varepsilon = h\nu$, $\boxed{p = \frac{\varepsilon}{c} = h\frac{\nu}{c} = \frac{h}{\lambda}}$

relativistic particles

$E = m_\gamma c^2 = \gamma m c^2$ total energy

$E = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\varepsilon(p)}_{\text{kinetic energy}}$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ Lorentz factor

$$\varepsilon(p) = E - mc^2 = \gamma mc^2 - mc^2$$

$$\Rightarrow \varepsilon(p) = (p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2$$

$$\Rightarrow p = \gamma m v$$

Blackbody Radiation (CH 3.2)

particles are photons

$\varepsilon_j = 0$

$\mu = 0$

$g = 2$ (spin-up , spin-down)

$\varepsilon(p) = pc$

$$\Rightarrow n(p) = \frac{1}{h^3} \frac{2}{\exp[\frac{pc}{kT}] - 1}$$

* state parameter n_γ : number density of photons

$$n_\gamma = \int_p 4\pi p^2 n(p)\, dp$$

$$n_\gamma = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp[\frac{pc}{kT}] - 1} \quad cm^{-3} \qquad (3.15)$$

* Here $T = T_{radi}$ radiation temperature

In LTE, $T_{rad} = T_{thermal}$

* n_γ : Let $x = \frac{pc}{kT} = \frac{\varepsilon(p)}{kT}$: dimensionless energy

$$\Rightarrow n_\gamma = 2\pi\, \zeta(3) \left(\frac{2kT}{ch}\right)^3 \approx 20.28\, T^3 \cdot cm^{-3} \quad (3.16)$$

$$\zeta(3) = \int_0^\infty \frac{x^2}{e^x - 1} dx = 1.2072$$

one Riemann Zeta function

* $P_\gamma = \frac{1}{3} \int_0^\infty n(p)\, pc\, 4\pi p^2\, dp$

$$P_\gamma = \frac{1}{3} a T^4 \quad dyne\ cm^{-2} \qquad (3.17)$$

$$a = \left(\frac{k^4}{c^3 h^3} \cdot \frac{8\pi^5}{15}\right) = 7.56 \times 10^{-15} \text{ erg cm}^{-3} K^{-4}$$

the radiation constant

also used $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$

* E_γ $\quad E_\gamma = \int_0^\infty n_{(p)}\, pc\, 4\pi p^2 dp$

$$E_\gamma = aT^4 \quad \text{erg cm}^{-3}$$

$\Rightarrow E_\gamma = 3P_\gamma$

γ-law equation of state : $P = (\gamma - 1)E$

$$\gamma = \frac{P}{E} + 1 = \frac{1}{3} + 1 = \frac{4}{3}$$

Therefore $\gamma = \frac{4}{3}$ for radiations

planck function :

— radiation intensity for a black body

$$\boxed{B_{\nu(T)} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}} \quad \text{erg cm}^{-2} s^{-1} Hz^{-1}$$

Can we derive $B_{\nu(T)}$? It is an experimental result

energy density in differential format (with ν)
u_ν frequency

$$E = \int_0^\infty u_\nu\, d\nu$$

$$\nu = \frac{\varepsilon}{h} = \frac{pc}{h} \Rightarrow d\nu = \frac{c}{h} dp$$

$$\Rightarrow dp = \frac{h}{c} d\nu$$

$$p = \frac{h}{c}\nu$$

$$\varepsilon_{(p)} = h\nu$$

$$E = \int n(p)\, \varepsilon(p)\, 4\pi p^2\, dp$$

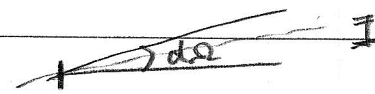
$$E = \int \frac{2}{h^3} \frac{1}{e^{h\nu/kT}-1} \cdot h\nu \cdot 4\pi \left(\frac{h}{c}\nu\right)^2 \cdot \frac{h}{c}\, d\nu = \int u_\nu\, d\nu$$

$$\Rightarrow u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1} \qquad \text{erg cm}^{-3}\ \text{Hz}^{-1}$$

differential energy density

radiation Intensity I: erg cm^{-2} s^{-1} std

$$dI = \frac{c\, du}{d\Omega\, ds}$$



energy across unit area unit std

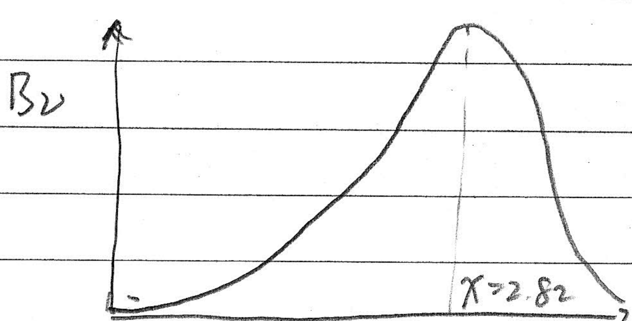
If uniform isotropic $\quad du \to u, \quad dI \to I$

$\qquad d\Omega \to 4\pi$

$$I = \frac{cu}{4\pi}$$

$$\Rightarrow I_\nu = B_\nu = \frac{c u_\nu}{4\pi} \qquad \text{---- (3-21)}$$

$$\Rightarrow B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}$$



⇐ Planck Distribution

$$x = \frac{h\nu}{kT} = \frac{\varepsilon(p)}{kT}$$

$$\frac{dB_\nu}{d\nu} = 0 \Rightarrow x_{max} = 2.82$$