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CH1. Preliminaries (11)

1.3. Virial Theorem & Application

\# Virial theorem : again, use (global) energy principles to understand the stars

$\omega = U + \Omega$.

How to relate U to Ω?

for a typical star, $\boxed{U = -\frac{\Omega}{2}}$ from Virial Theorem

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega} \quad \text{------} \quad (1.18)$$

I : moment of inertial $\left(\begin{matrix} F = ma \\ \tau = I\omega \end{matrix}\right)$

$I = mr^2$ for a single particle

$I = \sum_i m_i r_i^2$ for a system of particles

K : kinetic energy of a system of particles related to internal energy U

$K = \sum_i \frac{1}{2} m_i V_i^2$ $\quad \boxed{K = U \text{ if } \gamma = \frac{5}{3} \text{ for ideal gas}}$

\# Prove the Virial Theorem

Consider a system of particles $\sum_i$

for $\sum_i \vec{P_i} \cdot \vec{r_i}$: $\vec{P_i}$: momentum

$\vec{r_i}$: position vector

(1) $\frac{d}{dt}(\sum_i \vec{P_i} \cdot \vec{r_i}) = \frac{d}{dt}(\sum_i m_i \dot{\vec{r_i}} \cdot \vec{r_i}) = \frac{d}{dt} \sum_i \frac{1}{2}\frac{d}{dt}(m_i r_i^2)$

because $\frac{1}{2}\frac{d r_i^2}{dt} = \frac{1}{2} 2\vec{r_i} \cdot \frac{d\vec{r_i}}{dt} = \dot{\vec{r_i}} \cdot \vec{r_i}$

$\frac{d}{dt}(\sum_i \vec{P_i} \cdot \vec{r_i}) = \frac{1}{2}\frac{d}{dt^2}(\sum m_i r_i^2) = \frac{1}{2}\frac{d^2 I}{dt^2}$

(2) $$\frac{d}{dt}\sum_i \vec{P}_i\cdot\vec{r}_i = \sum_i\left(\frac{d\vec{P}_i}{dt}\right)\cdot\vec{r}_i + \sum_i \vec{P}_i\cdot\frac{d\vec{r}_i}{dt}$$

$$\frac{d\vec{P}_i}{dt} = \vec{F}_i$$: all forces acted on particle i

$$\sum_i \vec{P}_i\cdot\vec{V}_i = \sum_i m_i\vec{V}_i\cdot\vec{V}_i = \sum_i m_i V_i^2 = 2K$$

where $K = \sum_i\left(\frac{1}{2}m_i V_i^2\right) = \sum_i K_i$

Thus $$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i \vec{F}_i\cdot\vec{r}_i}\quad \text{--- } 1.17$$

$\boxed{\sum_i \vec{F}_i\cdot\vec{r}_i}$: Virial of clausius

* For a gravitational system, such as stars

$$\boxed{\text{Virial} = \Omega} = \sum_i \vec{F}_i\cdot\vec{r}_i$$

$$F_i = \sum_{\substack{j\\ j\neq i}} F_{ij} = \sum_{i<j} F_{ij} + \sum_{i>j} F_{ij}$$

$$\sum_i \vec{F}_i\cdot\vec{r}_i = \sum_i\left(\sum_{i<j} F_{ij}\cdot r_i + \underbrace{\sum_{i>j} F_{ij}\cdot r_i}\right)$$

$$\sum_{i>j} F_{ij}\cdot r_i \xrightarrow{\text{Flip } i,j} \sum_{i<j} F_{ji}\cdot r_j$$

$$\vec{F}_{ij} = -\frac{Gm_im_j}{r_{ij}^3}(\vec{r}_i-\vec{r}_j) \quad \text{and} \quad \vec{F}_{ji} = -\vec{F}_{ij}$$

$$\Rightarrow \sum_i \vec{F}_i\cdot\vec{r}_i = -\sum_{\substack{i,j\\ i<j}} \vec{F}_{ij}\cdot(\vec{r}_i-\vec{r}_j) = -\sum_{\substack{i,j\\ i<j}}\frac{Gm_im_j}{r_{ij}}$$

In discrete form, total

$$\Omega = -\sum_{\substack{i,j\\ i<j}}\frac{Gm_im_j}{r_{ij}}$$

* How about K

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i \vec{p}_i \cdot \vec{v}_i$$

From statistical mechanics. Pressure is

$$P = \frac{1}{3} \int_{\vec{p}} n(\vec{p})\, \vec{p} \cdot \vec{v}\, d^3\vec{p}$$

$d^3\vec{p}$: momentum space from $(0, \infty)$

$\frac{1}{3}$: from integration in all directions for isotropic distribution

P is defined by momentum transfer through a surface

In discrete form $P = \frac{1}{3} \sum_i \vec{p}_i \cdot \vec{v}_i$ in unit volume

$$\Rightarrow K = \frac{3}{2} \int P\, dV$$

$$2K = 3 \int P\, dV = 3 \int \frac{P}{\rho}\, dM_r$$

since $\rho = \dfrac{dM_r}{dV}$

$$\Rightarrow \boxed{\frac{1}{2} \frac{d^2 I}{dt^2} = \int_M \frac{3P}{\rho}\, dM_r + \Omega} \quad \text{--- (1.23)}$$

How to relate to $\mathcal{U}$

* γ-law equation of state (general form)

$$P = (\gamma - 1)\, \rho E$$

pressure (P); internal energy (E, erg g^{-1})

$\gamma = \frac{c_p}{c_v}$: ratio of specific heats

c_p : specific heat at constant pressure

c_v : " " " " volume

For ideal gas. $\rho E = n \cdot \frac{3}{2} kT$

(ρE: erg cm^{-3}), n: number density

$$\gamma = \frac{5}{3}$$

$$\boxed{P = \left(\tfrac{5}{3} - 1\right) n \tfrac{3}{2} kT = nkT}$$

reduce to ideal gas law

$$2K = \int \frac{3P}{\rho} dM_r = \int \frac{3}{\rho}(\gamma - 1)\rho E\, dM_r$$

$$= 3(\gamma - 1)\int E\, dM_r = 3(\gamma - 1)\,\mathcal{U}$$

If $\gamma = \frac{5}{3} \Rightarrow 2K = 2\mathcal{U} \Rightarrow K = \mathcal{U}$

Ideal gas star, total random kinetic energy = total internal energy

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \underline{3(\gamma - 1)\,\mathcal{U} + \Omega} \quad \text{--- } 1.25$$

$$W = \mathcal{U} + \Omega \Rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = 3(\gamma - 1) W - (3\gamma - 4)\Omega \quad \text{--- } 1.26$$

For Hydrostatic Equilibrium $d^2 I / dt^2 = 0$

$$\boxed{W = \frac{3\gamma - 4}{3(\gamma - 1)}\,\Omega} \quad \text{--- } 1.27$$

1.3.2 Virial Theorem Application

— Kelvin-Helmholtz Time Scale

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 3(\gamma-1)U + \Omega \qquad (1.25)$$

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 3(\gamma-1)W - (3\gamma-4)\Omega \qquad (1.26)$$

For a star in hydrostatic equilibrium.

$$\frac{1}{2}\frac{d^2 I}{dt} = 0$$

$$\Rightarrow \quad U = -\frac{1}{3(\gamma-1)}\Omega$$

$$W = \frac{3\gamma-4}{3(\gamma-1)}\Omega$$

For an ideal gas star, $\gamma = \frac{5}{3}$

$\Delta U = -\frac{1}{2}\Delta\Omega$: Half of gravitational energy goes to internal energy, the other half radiates away

$\Delta W = \frac{1}{2}\Delta\Omega$: Total energy is negative

Total energy changes responsible for luminosity.

$$L = -\frac{dW}{dt} = -\frac{1}{2}\frac{d\Omega}{dt} = +\frac{q}{2}\frac{d}{dt}\left(\frac{GM^2}{R}\right)$$

$$L = -\frac{q}{2}\frac{GM^2}{R}\left(\frac{d\ln R}{dt}\right)$$

Introduce the KH (Kelvin-Helmholtz) time scale: characteristic e-folding time for radius change

$$\frac{1}{\tau_{KH}} = -\frac{d\ln R}{dt}$$

$$d \ln R = -\frac{1}{\tau_{KH}} dt$$

$$\ln R \Big|_{R_0}^{R} = -\frac{1}{\tau_{KH}} \Big|_0^t$$

$$\ln R/R_0 = \frac{t}{\tau_{KH}}$$

$$\Rightarrow R = R_0 e^{-\frac{t}{\tau_{TH}}}$$ thus e-folding time

$$L = \frac{q}{2} \cdot \frac{GM^2}{R} t_{KH}$$

$$\Rightarrow t_{KH} \approx \frac{q}{2} \frac{GM^2}{LR} \quad \text{-----} \quad (1.31)$$

In solar units, $q = 3/2$

$$t_{KH} \approx 2 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{L}{L_\odot}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-1} \text{ years} \quad - 1.32$$

For the Sun, $t_{KH} = 2 \times 10^7$ years.

$\Rightarrow$ The Sun is not powered by gravitational contraction.

1.3.4 Virial Theorem Application

— Estimate stellar Temperature

$$\mathcal{U} = -\frac{\Omega}{2}$$

$$\mathcal{U} = \int E\, dV = \int \frac{E}{\rho}\, dM$$: E: internal energy per unit volume.

For ideal gas $E = \frac{3}{2} n k T$

$$n = \frac{\rho}{\mu} N_A$$

μ: mean molecular weight in amu

$N_A = 6.022 \times 10^{23}$ Mole^{-1}: Avogadro's constant

Assuming a star has a uniform temperature ("big" assumption)

$$\mathcal{U} = -\frac{\Omega}{2} = \int \frac{3}{2} \cdot \underbrace{\frac{\rho}{\mu} N_A}_{n} k T \underbrace{\frac{1}{\rho} dM}_{\Rightarrow M}$$

$$\Rightarrow \frac{G}{2} \frac{M^{\cancel{2}}}{R} = \frac{N_A}{2\mu} k T \cancel{M}$$

$$R = \left(\frac{M}{\rho}\right)^{\frac{1}{3}}$$

$$\Rightarrow T = 4.09 \times 10^6 \mu \left(\frac{M}{M_\odot}\right)^{2/3} \rho^{\frac{1}{3}} \text{ K} \quad (1.36)$$

The Sun's temperature $\sim 4 \times 10^6$ K

Realistically, $T_c \sim 15 \times 10^6$ K

T High enough to ignite nuclear fusion

1.3.3 : Virial Theorem Application
— A Dynamic Time Scale

Assuming internal pressure disappears. (e.g., photon-disintegration of iron core before Supernova, or inverse-β processes), a star collapses

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \Omega$$

Dimensional analysis: $I \approx MR^2$

$$\Omega = -\frac{GM^2}{R}$$

$$t \propto t_{dyn}$$

$$\Rightarrow \frac{I}{t_{dyn}^2} = \Omega$$

$$t_{dyn}^2 = \frac{I}{\Omega} = \frac{MR^2}{G\frac{M^2}{R}} = \frac{1}{G\frac{M}{R^3}} \approx \frac{1}{G\rho}$$

$$\boxed{t_{dyn} \approx \frac{1}{(G\langle\rho\rangle)^{\frac{1}{2}}}} \quad \text{--- } 1.33$$

$\langle\rho\rangle$ average density of a star

$\langle\rho_\odot\rangle = 1.4 \text{ g cm}^{-3}$

In solar unit $\boxed{t_{dyn} = \frac{0.04}{[\rho/\rho_\odot]^{\frac{1}{2}}} \text{ days}}$ --- 1.40

For the Sun, the collapse time $t_{dyn} \sim 1$ hour only

1.4 Constant - Density Model:

only assume density is constant: $\rho = \rho_c$
but not temperature and pressure

* a more realistic model.
* Find $P(r)$, $T(r)$

$$\frac{dP}{dM_r} = -\frac{G M_r}{4\pi r^4} \quad \text{--- --- 1.16}$$

Lagrangian form of hydrostatic equilibrium equation

$$M_r = \frac{4}{3}\pi r^3 \rho_c$$

$$M = \frac{4}{3}\pi R^3 \rho_c$$

$$\frac{r^3}{R^3} = \frac{M_r}{M}$$

$$\Rightarrow dP = -\frac{GM^2}{4\pi R^4}\left(\frac{M_r}{M}\right)^{-\frac{1}{3}} \frac{dM_r}{M}$$

$$P\Big|_0^{M_r} = -\frac{GM^2}{4\pi R^4}\left(\frac{1}{2/3}\right)\left(\frac{M_r}{M}\right)^{\frac{2}{3}}\Big|_0^{M_r}$$

At center, $r=0$, $M_r=0$, $P = P_c$ (maximum)
At surface $r=R$, $M_r = M$, $P=0$

$$P(r) - P_c = -\frac{3}{8\pi}\frac{GM^2}{R^4}\left(\frac{M_r}{M}\right)^{\frac{2}{3}}$$

$$\Rightarrow \boxed{P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}}$$

$$\Rightarrow P(r) = P_c\left[1-\left(\frac{M_r}{M}\right)^{\frac{2}{3}}\right] = P_c\left[1-\left(\frac{r}{R}\right)^2\right] \quad \text{--- 1.41}$$

$$P_c = 1.34\times 10^{15}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4} \text{ dyne cm}^{-2} \quad \text{--- (1.42)}$$

1.4.2 Temperature Distribution in constant ρ model

Ideal gas. $P = nkT = \frac{\rho}{\mu} N_A kT$

$$T = \frac{\mu}{N_A k} \frac{P}{\rho}$$

$$T_c = \frac{\mu}{N_A k} \cdot \frac{3}{8\pi} \frac{GM^2}{R} \cdot \frac{1}{\left(\frac{M}{\frac{4}{3}\pi R^3}\right)}$$

$$T_c = \frac{1}{2} \frac{GM}{R} \frac{\mu}{N_A k}$$

$$T_c = 1.15 \times 10^7 \mu \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} K \quad \text{--- (1.56)}$$

$$T = T_c \left[1 - \left(\frac{r}{R}\right)^2\right]$$

A more realistic model, higher T_c at center.

but not realistic at $r = R \Rightarrow T = 0$.

#1.7 Evolution lift times on the Main Sequence

Main sequence: 10% of H is used up

$$\Delta M = 0.1 M$$

$$\boxed{E = mc^2}$$

$$E = 1(g)\,(3 \times 10^{10}(cm/s))^2 = 9 \times 10^{20}\ ergs/g$$

Total nuclear energy generated

$$W = M \times \underbrace{0.1}_{(10\%)} \times \underbrace{0.007}_{(0.7\%)} \cdot 9 \times 10^{20}$$

$$t_{nuc} \simeq \frac{W}{L}$$

$$t_{nuc} \simeq 10^{10} \left(\frac{M}{M_\odot}\right)\left(\frac{L}{L_\odot}\right)^{-1} \text{ years}$$

For the Sun, $t_{nuc} \sim 10^{10}$ yrs

* From observations and simple dimensional analysis

$$\left(\frac{L}{L_\odot}\right) \sim \left(\frac{M}{M_\odot}\right)^{3.9}$$

$$\Rightarrow t_{nuc} \simeq 10^{10} \left(\frac{M}{M_\odot}\right)^{-2.9} \text{ years}$$

$M = 10 M_\odot$, $t_{nuc} = 1.25 \times 10^7$ yrs $= 12$ million yrs

$M = 25 M_\odot$ $t_{nuc} = 8.8 \times 10^5$ yrs $= 0.8$ million yrs

$M = 0.5 M_\odot$ $t_{nuc} = 7.46 \times 10^{10}$ yrs $= 74$ billion yrs

>> life of universe