CH1 : Preliminaries

[Lect 03, Sep. 11, 2012]

1.1 . Hydrostatic Equilibrium

* what is a star ?

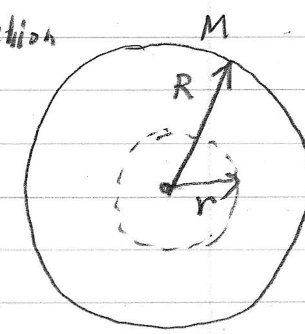
— From dynamic point of view, the star is a stable object in which self-gravitational force is balanced by internal pressure gradient.

Simply put : $\boxed{\rho_{(r)} \vec{g}_{(r)} = \frac{dP_{(r)}}{dr}}$

Called $\boxed{\text{Hydrostatic Equilibrium}}$ condition

* what is stellar structure ?

— theoretical understanding of the structure involves the knowledge of a set of parameters in the function of indepedent variable r



① $\rho_{(r)}$: density (g/cm^3)

② $T_{(r)}$: temperature (K)

③ $P_{(r)}$: pressure (dyne cm^{-2})

— stress tensor is isotropic ($\overleftrightarrow{P} \to P$)

M : total mass

R : total radius

④ $M_{(r)}$: mass within r (g)

⑤ $L_{(r)}$: luminosity (erg s^{-1})

⑥ $\varepsilon_{(r)}$: energy generate rate (erg g^{-1} s^{-1})

— nuclear reaction rate from microphysics (CH6)

(7) $\kappa(r)$: opacity ($cm^2\ g^{-1}$)
- regulate the radiation transfer (CH4)
- depends on photon scattering / absorption from microphysics (CH4)

* Need at least seven equations to solve the structure of a star

* (1) P, T, ρ : state parameters

(2) ε, κ : parameters of microphysics, but depends on state parameters

* $\varepsilon = \varepsilon_0 \rho^{\lambda} T^{\nu}$ ------ (1.59)

nuclear reaction $\propto \rho$, $\propto T$

* $\kappa = \kappa_0 \rho^{n} T^{-s}$ ---- (1.62)

opacity $\propto \rho$, $\propto \frac{1}{T}$

(3) M, L, R : global parameters

$M_\odot = 1.99 \times 10^{33}$ g

$L_\odot = 3.85 \times 10^{33}$ erg s^{-1}

$R_\odot = 6.96 \times 10^{10}$ cm

1.1 † (Extended): Equations of stellar structure

* Seven parameters need seven equations

Equation of mass conservation		× 1
Equation of momentum conservation		× 1
Equation of Energy Conservation		× 1
microphysics equations:	radiation transfer	× 1
	nuclear reaction	× 1
	opacity	× 1
Equation of state		× 1

1. Equation of Mass Conservation

* General form: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

or $\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0$

$\frac{\partial \rho}{\partial t}$: local derivative,

$\frac{d\rho}{dt}$: total derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$

In a star, static $\Rightarrow \frac{\partial \rho}{\partial t} = 0$

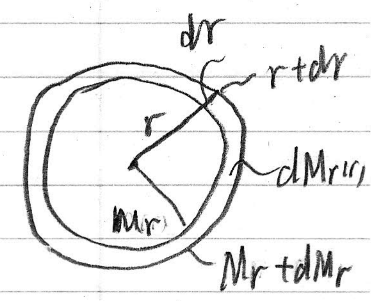
stable $\Rightarrow \vec{v} = 0$

$$\rho_{(r)} = \frac{dM_r(r)}{4\pi r^2 dr}$$

or $dM_r = 4\pi r^2 \rho_{(r)} dr$ — shell mass

$$M_R = \int_0^R 4\pi r^2 \rho_{(r)} dr$$

---- (1.2)



mass shell model (spherical symmetry)

2. Equation of Momentum Conservation

* general form : $\rho \frac{d\vec{v}}{dt} = -\nabla P + \rho \vec{g}$

static / steady : $\frac{d\vec{v}}{dt} = 0$

$\Rightarrow \nabla P = \rho \vec{g}$ (hydrostatic Equilibrium)

$$g(r) = \frac{G M_r(r)}{r^2}$$

$$G = 6.67 \times 10^{-8} \ g^{-1} \ cm^3 \ s^{-2}$$

In solar unit $g(r) = 2.74 \times 10^4 \left(\frac{M}{M_\odot}\right) \left(\frac{r}{R_\odot}\right)^{-2} cm \ s^{-2}$.

$g = 2.74 \times 10^4 \ cm \ s^{-2}$ on the surface of Sun

$$\boxed{\frac{dP(r)}{dr} = -\frac{G M_r(r)}{r^2} \rho(r)} \quad \text{------ (1.6)}$$

* Lagrangian format $\left(\frac{dP(r)}{dM_r(r)} = ?\right)$ of hydrostatic equilibrium

Since $\frac{dM_r}{dr} = 4\pi r^2 \rho(r)$

$$\frac{dP(r)}{dM_r(r)} = \frac{dP}{dr} \cdot \frac{dr}{dm} = -\frac{G M_r}{r^2} \rho(r) \frac{1}{4\pi r^2 \rho(r)}$$

$$\Rightarrow \boxed{\frac{dP(r)}{dM_r(r)} = -\frac{G M_r(r)}{4\pi r^2}} \quad \text{------ (1.6b)}$$

3. Equation of Energy

general form:

$$\rho \frac{de}{dt} + P\nabla\cdot\vec{V} = \nabla\cdot(\vec{K}\cdot\nabla T) + Q_{source} - Q_{sink}$$

e: internal energy density

$\nabla\cdot(\vec{K}\cdot\nabla T)$: thermal conduction

In a star, reduce to

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\,\varepsilon(r) \quad \text{------} (1.57)$$

(In the nuclear reaction layer)

$$\frac{dL(r)}{dr} = 0 \quad \text{(In radiation layer)}$$

4. Microphysics Equation of radiation transfer

$$L(r) = -\frac{4\pi r^2 c}{3K\rho}\frac{d(aT^4)}{dr} \quad \text{---} (1.60)$$

(Fick's Law)

5. Microphysics Equation of nuclear reaction

$$\varepsilon = \varepsilon_0 \rho^{\lambda} T^{\nu} \quad \text{---} (1.59)$$

H → He PP chain: $\lambda = 1$, $\nu = 4$

H → He CNO cycle: $\lambda = 1$, $\nu = 15$

He → C Triple α: $\lambda = 2$ $\nu = 40$

6 Microphysics of opacity

$$K = K_0 \rho^n T^{-s} \quad \text{----} (1.62)$$

(Kramer's opacity)

Free electrons: $n = 0$, $s = 0$

Atoms $n = 1$, $s = 3.5$

7. Equation of state (parameters)

$$P = P(\rho, T) = P_0 \rho^{\chi_\rho} T^{\chi_T} \quad \cdots \quad (3.87)$$

(usual power-law format)

For ideal gas : $\chi_\rho = 1 \quad \chi_T = 1$

$$P = nkT$$

For radiation photons : $\chi_\rho = 0, \ \chi_T = 4$

$$P = \frac{1}{3} a T^4$$

k, Boltzmann constant : 1.38×10^{-16} erg K^{-1}

a : radiation constant $= 7.56 \times 10^{-15}$ erg $cm^{-3} K^{-4}$

* What about evolution ?

$\frac{d}{dt} = 0$. evolution is quasi-static

what changes ? composition $\mu(t)$

(70% H, 30% He) → (100% He)

$\Rightarrow \rho(t), T(t), P(t), \Rightarrow \varepsilon, K$

$\Rightarrow L_R(t), R(t)$: (total luminosity and Size changes)

but $M(t)$ is a conserved parameter

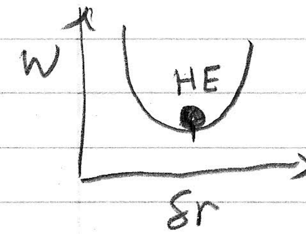
CH1 : Preliminaries

1.2. An Energy Principle

* From energy principle to derive hydrostatic equilibrium equation

$$\boxed{\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^2}} \qquad (1.16)$$

* Energy quantities:



Ω : gravitational energy

U : internal energy

$W = \Omega + U$: total energy of star

In a global scale ($r = R$), energy generation rate and/or dissipation rate L is negligible.

Thus, in principle, perturbation of star will not change the total energy

$$(\delta W)_{ad} = 0$$

since the star is in hydrostatic equilibrium

$(\ \ )_{ad}$: adiabatic, no heat transfer

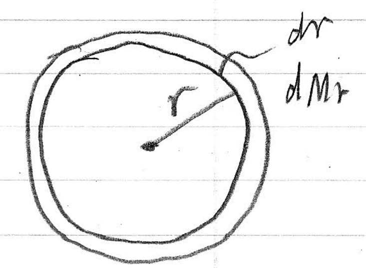
* Ω : gravitational energy
  - energy released to assemble M into R from infinity
  - OR, energy needed to disperse M into infinity

$\Omega = 0$ all dispersed

$\Omega < 0$ assembled, contracted to a star

* move shell $dM(r)$ from r to ∞

$$d\Omega = \int_r^\infty F_g dr' = -\int_r^\infty \frac{G M_r dM_r}{r'^2} dr'$$



$$\boxed{d\Omega = -\frac{G M_r dM_r}{r}}$$

gravitational energy in mass shell dM_r

$$\boxed{\Omega = \int d\Omega = -\int_0^M \frac{G M_r}{r} dM_r} \quad \text{---- (1.7)}$$

$$\Omega = -q \frac{GM^2}{R} \quad \text{------ (1.8)}$$

q is a constant depending on density distribution

$q = \frac{3}{5}$ if uniform density

* ρ is uniform

$$M_r = \frac{4}{3}\pi r^3 \rho \quad , \quad \rho = \frac{3M_r}{4\pi r^3}$$

$$dM_r = 4\pi r^2 \rho \, dr$$

$$\Omega = -\int_0^R \frac{G \cdot \frac{4}{3}\pi r^3 \rho}{r} \cdot 4\pi r^2 \rho \, dr$$

$$\Omega = -\frac{16\pi^2}{3} \rho^2 G \int_0^R r^4 dr$$

$$= -\frac{16}{15} \pi^2 \rho^2 R^5 G$$

$$\Rightarrow \boxed{\Omega = -\frac{3}{5}\frac{GM^2}{R}} \quad \text{for } \rho = \text{const}$$

$\Omega_\odot \approx 3.8 \times 10^{48}$ erg

* $\delta\Omega$. perturbation only on the position, not the mass

$$\delta\Omega = -\delta \int_0^M \frac{GM_r}{r}\, dM_r = -\int_0^M \left(\frac{1}{r+\delta r} - \frac{1}{r}\right) GM_r\, dM_r$$

$$\frac{1}{r+\delta r} = \frac{1}{r}\frac{1}{(1+\frac{\delta r}{r})} = \frac{1}{r}(1+\frac{\delta r}{r})^{-1}$$

$$= \frac{1}{r}(1-\frac{\delta r}{r}) = \frac{1}{r} - \frac{\delta r}{r^2}$$

$$\Rightarrow \delta\Omega = \int_0^M \frac{GM_r}{r^2}\, \delta r\, dM_r$$

* U: total internal energy:
  random kinetic energy of particles
  or thermal energy of particles

$$U = \int_0^M E\, dM_r \qquad (1.6)$$

E: local mass-specific internal energy density ($\mathrm{erg\ g^{-1}}$)

Internal energy density sometimes in volume $\mathrm{cm^{-3}}$

E (volume-specific) = E (mass-specific) $\times \rho$

$$\delta U = \int_0^M \delta E\, dM_r$$

* First Law of thermal dynamics

$$dQ = dE + P\, dV_\rho$$

heat ($\mathrm{erg\ g^{-1}}$) — internal energy ($\mathrm{erg\ g^{-1}}$) — ($\mathrm{erg\ g^{-1}}$) work done by gas on the system

$V_\rho = \frac{1}{\rho}$ ($\mathrm{cm^3\ g^{-1}}$): specific volume

Adiabatic assumption $dQ = 0$

$$\Rightarrow dE = -P\, dV_\rho$$

$$\delta E = -P\,\delta V_\rho$$

$$V_\rho = \frac{1}{\rho} = \frac{4\pi r^2 dr}{dM} = \frac{d\left(\frac{4\pi r^3}{3}\right)}{dM_r}$$

$$\delta V_\rho = \delta\left(\frac{d\,\frac{4\pi r^3}{3}}{dM_r}\right) = \frac{d(4\pi r^2 \delta r)}{dM_r}$$

$$\delta U = -\int_M P\,\frac{d(4\pi r^2 \delta r)}{dM_r}\,dM_r$$

Integration by parts

$$\delta U = -P \cdot 4\pi r^2 \delta r\Big|_0^M + \int 4\pi r^2 \delta r\,dP$$

At center, $\delta r = 0$. center is not allowed to move in spherical geometry

At surface $M_r = M_R$, "zero-boundary condition for pressure", $P = 0$

$$\delta U = \int 4\pi r^2 \delta r\,\frac{dP}{dM}\,dM$$

$$\star\ (\delta W)_{ad} = (\delta\Omega) + (\delta U)_{ad}$$

$$= \int_M \Big[\underbrace{\frac{GM_r}{r^2} + 4\pi r^2\,\frac{dP}{dM_r}}_{=0}\Big]\,\delta r\,dM_r$$

hydrostatic equilibrium $(\delta W) = 0$ for any δr

$\Rightarrow$ integrant $= 0$

$$\Rightarrow \frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^2} \qquad (1.16)$$