2011 April 13

Mixing Length Theory (MLT) (CH 5.1)

How to quantify the turbulent convection, and determine the convection flux.

MLT: to simplify the process, assuming the convection parcel travels a characteristic time "ℓ", the mixing length, before it lose identity as the parcel break-up and merge with the surrounding fluid.

A typical "ℓ" is a fraction of λ_P (pressure scale height)

We need to solve a time dependent energy equation and momentum equation

because $v \neq 0$.

Energy Equation: Radiative Leakage (CH 5.1.2)

Assuming non-adiabatic, as bubble rises, heat transfer cools the bubble, thus reduce the strength of convection

If non-heat transfer, or adiabatic, $\Rightarrow$ super-convection

$$\frac{dQ'}{dt} = -\nabla \cdot F_{rad} \qquad \boxed{\rho', T'} \quad \rho, T$$

$$dQ' = C_P \rho \, dT'$$

$$C_P - C_V = \frac{P}{\rho T} \frac{\chi_T^2}{\chi_\rho^2} \qquad (3.90)$$

$$C_P - C_V = \frac{N_A k}{\mu} \quad \text{(ideal gas)} \qquad (3.40)$$

$$C_V = \frac{3 N_A k}{2\mu} \left(\frac{8 - 7\beta}{\beta} \right) \qquad (3.105)$$

$$C_V = \frac{3 N_A k}{2\mu} \quad \text{(ideal gas)} \qquad \boxed{\text{erg g}^{-1}\,\text{K}^{-1}}$$

$$C_P = \frac{5}{2} \frac{N_A k}{\mu} \quad \text{(ideal gas)}$$

(Continued).

$$\left(\frac{\partial T'}{\partial t}\right) = -\frac{1}{\rho C_p} \nabla \cdot F_{rad} \qquad \text{---} \quad (5.12)$$

$$F_{rad} = -K \nabla T \qquad \text{---} \quad (5.13)$$

$$K = \frac{4acT^3}{3\kappa\rho} \qquad \text{---} \quad (5.14)$$

K: (radiation) diffusion constant

$$\left(\frac{\partial T'}{\partial t}\right) = \frac{K}{\rho C_p} \nabla^2 T = \frac{4acT^3}{3\kappa\rho^2 C_p} \nabla^2 T \qquad \text{---} \quad (5.15)$$

$$\nu_T = \frac{4acT^3}{3\kappa\rho^2 C_p} \qquad \text{---} \quad (5.16)$$

ν_T: thermal diffusivity (due to radiation) ; cm^2/s

$$\left(\frac{\partial T'}{\partial t}\right) = \nu_T \nabla^2 T \qquad \text{---} \quad (5.17)$$

Usually in MLT: $\boxed{\nabla^2 T = \frac{T - T'}{\ell^2}}$

$\frac{\partial T'}{\partial t}$: Local derivative at a fixed point; Eulerian form

Need to consider the movement of the parcel

$\frac{DT'}{dt} = \frac{dT'}{dt}$: total derivative; following the parcel
: Lagrangian form

$$\frac{DT'}{Dt} = \left(\frac{\partial T'}{\partial t}\right) + \vec{w} \cdot \nabla T' \qquad \text{---} \quad (5.18)$$

w: the parcel velocity

$$\underbrace{\vec{w} \cdot \nabla T'}_{} = w \frac{\partial T'}{\partial z} = w \underbrace{\left(\frac{dT'}{dz}\right)_{ad}}_{} = -w \beta_{ad}$$

decribe T' change without heat loss

$$\frac{DT'}{Dt} = \frac{\nu_T}{\ell^2}(T - T') - \beta_{ad} w \qquad \text{---} \quad (5.19)$$

$$\Delta T = T' - T \quad \text{------} \quad (5.20)$$

$$\frac{DT}{Dt} = \frac{dT}{dz}\frac{dz}{dt} = -\beta w$$

adiabatic — leaking

$$\boxed{\frac{D\Delta T}{Dt} = \frac{DT'}{Dt} - \frac{DT}{Dt} = (\beta - \beta_{Ad})w - \frac{\nu_T}{\ell^2}\Delta T} \quad \text{------} \quad (5.21)$$

This describes how temperature difference changes with time

① depend on how temperature changes with z

② depend on how radiation energy transfers

Equation of Motion (CH 5.1.3)

Buoyancy force

$\rho' \rho \quad p' = p$

$$\rho \frac{dw}{dt} = (\rho - \rho')g$$

$$\frac{dw}{dt} = \frac{(\rho - \rho')}{\rho} g \quad \text{------} \quad (5.22)$$

* Q: coefficient of thermal expansion $\quad Q > 0$

$$-Q \equiv \left(\frac{d\ln\rho}{d\ln T}\right)_P = -\frac{\chi_T}{\chi_\rho} \quad \text{------} \quad (5.25)$$

$$d\ln p = \chi_\rho \, d\ln\rho + \chi_T \, d\ln T$$

$$Q = -\frac{\Delta\rho}{\rho} \cdot \frac{T}{\Delta T} \quad ; \quad Q = 1 \text{ for ideal gas}$$

$$\frac{\rho' - \rho}{\rho} = \frac{\Delta\rho}{\rho} = -Q\frac{\Delta T}{T} = -Q\frac{T' - T}{T}$$

$$\boxed{\frac{dw}{dt} = \frac{Qg}{T}\Delta T} \quad \text{------} \quad (5.26)$$

5.21 and 5.26 combined to determine the convection

$\Rightarrow w(t)$

$\Rightarrow \Delta T(t)$

Convective Efficiencies and Time Scales (CH 5.1.4)

A simple instability analysis of energy equation + momentum Eq.

$$\frac{D(\Delta T)}{Dt} = (\beta - \beta_{ad}) w - \frac{\nu_T}{l^2}\Delta T$$

$$\frac{dw}{dt} = \frac{Qg}{T}\Delta T$$

σ: growth rate

σ: real — positive: grow; negative: damping

$$\Delta T = T_0 e^{\sigma t}$$

$$w = w_0 e^{\sigma t}$$

σ: imaginary, oscillation

$$\underbrace{\frac{D^2(\Delta T)}{Dt}}_{\frac{T_0\sigma^2 e^{\sigma t}}{\sigma^2 \Delta T}} = (\beta - \beta_{ad}) \underbrace{\frac{dw}{dt}}_{\frac{Qg}{T}\Delta T} - \frac{\nu_T}{l^2} \underbrace{\frac{D(\Delta T)}{Dt}}_{\frac{T_0 e^{\sigma t}\sigma}{\Delta T}}$$

$$\sigma^2 + \sigma \frac{\nu_T}{l^2} - \frac{Qg}{T}(\beta - \beta_{ad}) = 0 \qquad \text{--- (5.27)}$$

Note: ν_T, l, Q, g, T, β, β_{ad} - all known

This quadratic equation can be solved easily.

* Brunt - Väisälä frequency N

$$N^2 = -\frac{Qg}{T}(\beta - \beta_{ad}) \qquad \text{---- (5.28)}$$

$$\sigma^2 + \underbrace{\sigma\frac{\nu_T}{l^2}}_{\text{damping term}} + N^2 = 0 \qquad \text{--- (5.30)}$$

grow or ~~decay~~ oscillate

$$\sigma = \frac{-\frac{\nu_T}{l^2} \pm \sqrt{\left(\frac{\nu_T}{l^2}\right)^2 - 4N^2}}{2} \; ; \qquad \sigma^2 = -N^2 \text{ (ignore damping)}$$

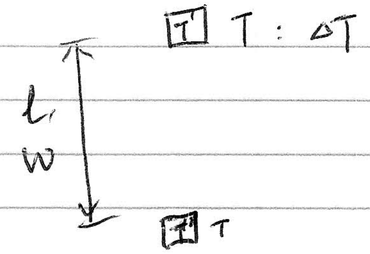
$N^2 < 0 \Rightarrow \beta - \beta_{ad} > 0 \Rightarrow$ Convection ; σ: real

$N^2 > 0 \Rightarrow \beta - \beta_{ad} < 0 \Rightarrow \sigma$: imaginary : oscillation.

N: grow rate and/or oscillation frequency (s^{-1})

Convective Flux (CH 5.1.5)

$$F_{conv} = \underbrace{\rho C_p \Delta T}_{\text{energy carried}} \cdot \underbrace{w}_{\text{how fast}} \quad \text{---- (5.36)}$$



$$w = \sigma l$$

$$\frac{d\Delta T}{dt} = \underbrace{(\beta - \beta_{ad})}_{\frac{\sigma^2 T}{gQ}}\, \underbrace{w}_{\sigma l} - \frac{\nu_T}{l^2}\cancel{\Delta T}$$

$$\sigma \Delta T$$

$$\sigma^2 = -N^2 = \frac{Q}{T} g (\beta - \beta_{ad}) \quad \text{-- (5.25)}$$

$$\Delta T = \frac{\sigma^2 l T}{Q g} \quad \text{---- (5.35)}$$

$$F_{conv} = \frac{\rho C_p T \sigma^3 l^2}{Q g} \quad \text{---- (5.36)}$$

$$L_{conv} = 4\pi r^2 F_{conv} \quad \text{---- (5.37)}$$

For adiabatic convection, or super-convection

$$\sigma = \sqrt{-N^2} = \sqrt{\frac{Q g}{T}(\beta - \beta_{ad})}$$

$$F_{conv} = \frac{\rho C_p l^2 T g^{\frac{1}{2}} Q^{\frac{1}{2}} (\nabla - \nabla_{ad})^{3/2}}{\lambda_p^{3/2}} \quad \text{--- 5.38}$$

Stellar Energy Sources (CH 6)

-Self - Study-

Energy generate rate ε : ergs $g^{-1} s^{-1}$

Stellar Modeling (CH 7)

- putting all the physics in "analytic" format together, then find the "exact" solution

Equations of stellar structure (CH 7.1)

① Conservation of mass

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad \text{or} \quad \frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho} \quad \cdots (7.5)$$

② Conservation of Momentum

$$\frac{dP}{dr} = -\rho g \quad \text{or} \quad \frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \quad - (7.6)$$

③ Conservation of Energy

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \quad \text{or} \quad \frac{dL_r}{dM_r} = \varepsilon \quad \cdots (7.7)$$

To find M_r, $\rho(r)$, $P(r)$, $T(r)$, $L(r)$,

need two more equations

④ Equation of state. ← from statistical mechanics

$$P = P(\rho, T, X) = P_0 \rho^{\chi_\rho} T^{\chi_T}$$

⑤ Energy diffusion equations ← from radiation transfer

$$\nabla_{rad} = \frac{3}{16\pi ac} \frac{P\kappa}{T^4} \frac{L_r}{GM_r} \quad \text{if } (\nabla_{rad} < \nabla_{ad}) \quad (7.8)$$

or. $\nabla = \nabla_{ad}$ (convection) if $\nabla_{rad} > \nabla_{ad}$ --(7.10)

And constituent equations from microphysics

⑥ $\kappa = \kappa(\rho, T, X)$: ($cm^2\, g^{-1}\, s^{-1}$) absorption coefficient

atomic physics ← nuclear physics

⑦ $\varepsilon = \varepsilon(\rho, T, X)$: ($ergs\, g^{-1}\, s^{-1}$) energy generation rate

Numerical solution

solving three-coupled first-order ordinary differential equations

It is trivial for a modern computer.

but, one has to specify the boundary condition.

1. $r = 0$, $M_r = 0$ or at $M_r = 0$, $r = 0$
2. $r = 0$, $L_r = 0$ or at $M_r = 0$ $L_r = 0$
3. $r = R$, $P = 0$ or at $M = M_*$, ~~T_{eff}~~ $P = 0$

but, what is R?

Numerical calculation (CH 7.2.2)

1. Marching Integration Method.

   Initial value ~~program~~ problem, integrate forward from the center, until $\rho = 0$ - the surface.

   Use "Runge-Kutta" method for integration

2. fitting method:

   Integration from both ends, joint at a point

3. Finite difference method

   Also "Newton-Raphson" or "Henyey" method

   solving equations ~~of~~ on a pre-constructed grid.

   On each grid point, a set of linear equations

   — solving a big matrix