2011 March 31

Radiative Transfer Equation – Recap

$$\mu \frac{dI}{d\tau} = I - S$$

To be complete

$$\mu \frac{d I_\nu(\tau, \mu)}{d\tau_\nu} = I_\nu(\tau, \mu) - S_\nu(\tau, \mu) \quad \text{--- (4.10)}$$

$$d\tau_\nu = -\kappa\rho\, dz \quad ; \quad dz = \mu\, ds = \cos\theta\, ds$$

$$S_\nu = \frac{j_\nu}{\kappa_\nu} : \text{source function}$$

General solution

$$I_\nu(\tau, \mu) = e^{-(\tau_0 - \tau)/\mu} I(\tau_0, \mu) + \int_\tau^{\tau_0} e^{-(t-\tau)/\mu} \frac{S(t)}{\mu} dt \quad \text{--- (4.11)}$$

Stellar interior

$$\boxed{S(t) = B(\tau) + (t - \tau)\left(\frac{\partial B}{\partial \tau}\right)_\tau} \quad \text{--- (4.14)}$$

$$I(\tau, \mu \geq 0) = B(\tau) + \mu\left(\frac{\partial B}{\partial \tau}\right)_\tau \quad \text{--- (4.15)}$$

$$I(\tau, \mu \leq 0) \approx B(\tau) + \mu\left(\frac{\partial B}{\partial \tau}\right)_\tau \quad \text{--- (4.16)}$$

$$F(\tau) = 2\pi \int_{-1}^{1} I(\mu)\, \mu\, d\mu$$

$$\boxed{F(\tau) = \frac{4\pi}{3} \frac{\partial B(\tau)}{\partial \tau}} \quad \text{--- (4.17)}$$

$$L(r) = 4\pi r^2 F(r) \quad \text{--- (4.18)}$$

Measure of Radiation Anisotropy

$$\text{Anisotropy} = \frac{F}{I} = \frac{\text{net flux}}{\text{Intensity}} = \frac{\frac{4\pi}{3}\frac{\partial B}{\partial \tau}}{B(\tau)}$$

Example. the Sun

$$L = 4\pi R^2 F \qquad L_\odot = 4\times 10^{33}\ \mathrm{erg\ s^{-1}}$$

$$R_\odot = 6.96\times 10^{10}\ \mathrm{cm}$$

$$F = \frac{L}{4\pi R^2} = \frac{4\times 10^{33}}{4\pi\times(6.96\times 10^{10})^2} = 2\times 10^{10}\ \mathrm{erg\ cm^{-2}\ s^{-1}}$$

$$B(\tau) = \frac{\sigma T_0^4}{\pi} \qquad T_0 \sim 10^7\ \mathrm{K}$$

$$\sigma = 5.67\times 10^{-5}\ \mathrm{erg\ cm^{-2}\ K^{-4}\ s^{-1}}$$

$$= 2\times 10^{23}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ sr^{-1}}$$

$$\text{Anisotropy} = \frac{2\times 10^{10}}{2\times 10^{23}} = 10^{-13}.$$

Therefore, in stellar interior, the luminosity is caused by a "small" anisotropy of the radiation field, which is further caused by the temperature gradient.

The Diffusion Equation (CH 4.2)

— radiation flow in classical diffusion format

$$F = \frac{4\pi}{3} \frac{\partial B(\tau)}{\partial \tau}$$

$$\Rightarrow F = -D_{rad} \frac{dT}{dr} \longleftarrow \boxed{\text{Fick's Law of Diffusion}}$$

D_{rad}: Diffusion coefficient of radiation

Diffusion: from area of high density to area of low density

$$\frac{\partial B(\tau)}{\partial \tau} = \frac{\partial B(T)}{\partial T} \frac{\partial T}{\partial \tau}$$

$$B(T) = \frac{\sigma T^4}{\pi} = \frac{c}{4\pi} a T^4 \; ; \quad \frac{\partial B(T)}{\partial T} = \frac{c}{\pi} a T^3$$

$$\partial \tau = -\kappa \rho \, \partial r$$

$$\Rightarrow F(r) = -\frac{4ac}{3} \frac{1}{\kappa\rho} T^3 \frac{dT}{dr} \quad \text{--- (4-23)}$$

~~F(r)~~ Therefore $\boxed{D_{rad} = \frac{4ac}{3} \frac{1}{\kappa\rho} T^3}$

Another form: $F(r) = -\frac{c}{3\kappa\rho} \frac{d(aT^4)}{dr}$

$u = aT^4$: radiation energy density

$\frac{1}{\kappa\rho} = \lambda$: photon mean free path

$$\boxed{F(r) = -\frac{1}{3} c \lambda \frac{du}{dr}}$$

$u = C_v T$: For gas

$$\boxed{F(r) = -\frac{1}{3} C_v c \lambda \frac{dT}{dr}}$$

which is similar to

classical thermal conduction formula

$$F = -\frac{1}{3} C_v V_e \lambda \frac{dT}{dr}$$

: V_e: mean electron velocity

λ: mean electron free path