March 23, 2011

CH4: Radiative and Conductive Heat Transfer
— About the energy equation

The context

① mass conservation : $\frac{dM_r}{dr} = 4\pi r^2 \rho(r)$

② Momentum conservation : $\frac{dP(r)}{dr} = -\rho(r) g(r)$

③ Energy conservation $\frac{dT(r)}{dr} = ?$

④ Equation of state : $P(r) = P_0 \rho(r)^{\chi_\rho} T(r)^{\chi_T}$

① + ② + ④ $\overbrace{\text{: Four variables}}^{\text{three equations,}}$. M_r, $P(r)$, $\rho(r)$, $T(r)$

Therefore, they can be solved if any can be specified.

Exp. Assume constant density $\longrightarrow \rho, P, T, M(r)$

Assume linear density $\longrightarrow \rho, P, T, M(r)$

However, for self-consistent solution, ③ energy equation needed

In numerical simulation, energy equation is always the most difficult, but critical.

$$\frac{dT(r)}{dr} = ?$$

$L = -D_{rad} \frac{dT}{dr}$: radiation $\boxed{\text{diffusion}}$ equation ~~in the envelope~~

$\frac{dL}{dr} = 4\pi r^2 \rho(r) \varepsilon(r)$: energy generation in the core

$\frac{dL}{dr} = 0$, $L = L_*$: constant luminosity in the envelope

D_{rad}, radiation diffusion coefficient — CH4

$\varepsilon(r)$: $\varepsilon_0 \rho^{\lambda} T^{\nu}$: nuclear energy generation rate — CH5

CH 4.1. Radiative Transfer

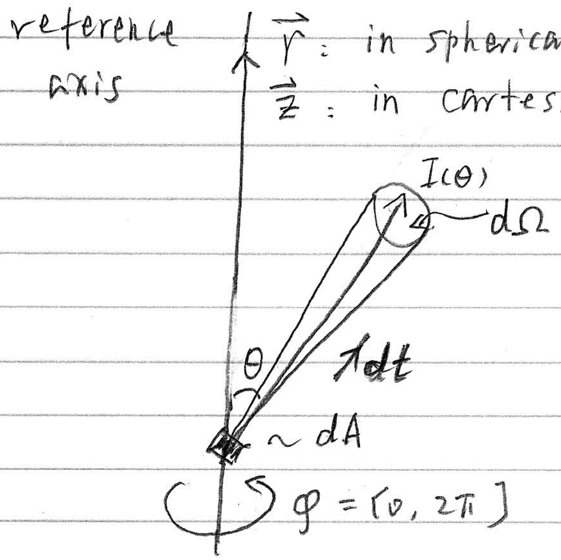
The theory about radiation energy : creation, absorption and <u>transfer</u>

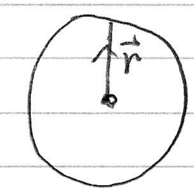
* $I(\theta)$: the specific radiation intensity

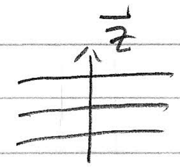
$$I(\theta) = \frac{\Delta E}{\Delta t \cdot \Delta A \cdot \Delta\Omega} = \frac{dE}{dt\, dA\, d\Omega} ; \quad \underline{erg \cdot s^{-1}\, cm^{-2}\, sr^{-1}}$$

$\vec{r}$: in spherical coordinate

$\vec{z}$: in Cartesian coordinate, or plane-parallel geometry







sphere $\vec{r}$; surface $\vec{z}$

φ : azimuthal angle ; spherical symmetry : $\frac{\partial}{\partial\varphi} = 0$

θ : colatitude ; isotropic : $\partial/\partial\theta = 0$

In stars, nearly isotropic ; small anisotropy $\Rightarrow I(\theta)$

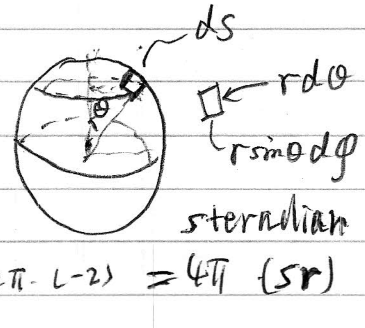
<u>radiation intensity</u> : radiation energy (dE) passing through a unit area (dA) along a <u>specific</u> direction (θ) in a unit time (dt) and within a unit solid angle ($d\Omega$)

* $d\Omega = \frac{ds}{r^2}$ or $ds = r^2 d\Omega$

$$d\Omega = \frac{r\,d\theta \cdot r\sin\theta\, d\varphi}{r^2} = \sin\theta\, d\theta\, d\varphi$$

$$\int d\Omega = \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\varphi = 2\pi \int_0^{\pi} \sin\theta\, d\theta$$

$$= -2\pi \int_1^{-1} d\cos\theta = -2\pi \cos\theta \Big|_1^{-1} = -2\pi \cdot (-2) = 4\pi \ (sr)$$



steradian

\# Derivatives from specific intensity $I(\theta)$

\# radiation energy density

From CH3, in $\underline{LTE}$, $u_{rad} = aT^4$ erg cm^{-3}

→ Black body radiation

$$du = \frac{I(\theta)\, d\Omega}{c} \quad : \text{energy in cone } d\Omega \text{ crossing unit area in unit length}$$

$$= \frac{dE}{dt\, dA\, \cancel{d\Omega}} \cdot \frac{\cancel{d\Omega}}{c} = \frac{dE}{dA \cdot \underbrace{(c\,dt)}_{ds}}$$

$(c\,dt)$ – unit length

ds – length traveled

= energy in cone $d\Omega$ in unit volume

$$u = \int du = \int_{4\pi} \frac{I(\theta)}{c} \cdot d\Omega = \int_0^{\pi} \frac{I(\theta)}{c} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$\mu = \cos\theta$, $-1 \le \mu \le 1$

$$u = \frac{2\pi}{c} \int_0^{\pi} I(\theta) \sin\theta\, d\theta = \frac{2\pi}{c} \int_{-1}^{+1} I(\mu)\, d\mu \quad \text{--- (4.2)}$$

If isotropic, $I(\mu) = I$

$$u = \frac{2\pi}{c} I \int_{-1}^{+1} d\mu = \frac{4\pi}{c} I$$

$$u = aT^4$$

$$\Rightarrow I = \frac{c}{4\pi} aT^4 = \frac{\sigma T^4}{\pi} = B(T) \quad \text{in LTE --- (4.6)}$$

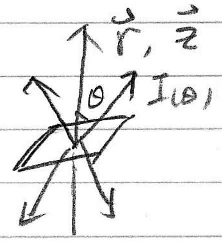
a: radiation constant

σ: Stefen-Boltzmann constant $\sigma = \frac{ca}{4}$

$B(T)$: integrated Planck Function

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad ; \quad B = \int_0^{\infty} B_\nu(T)\, d\nu \quad \text{--- (4.7)}$$

⇒ Radiation intensity in LTE is integrated planck function



radiation flux F:

net radiation energy across a unit area.

It is related to luminosity : $L = 4\pi r^2 F$

$$dF = \underbrace{I(\theta)\cos\theta}\, d\Omega$$

radiation energy cross the "surface" with $\vec{r}$ as the normal direction in cone $d\Omega$

If $\theta = \frac{\pi}{2}$, $I(\theta) \perp \vec{r}$. $dF = 0$, no crossing

$$F = \int dF = \int_{4\pi} I(\theta) \underbrace{\cos\theta}_{\mu} \underbrace{\sin\theta\, d\theta}_{-d\mu}\, d\varphi$$

$$F = 2\pi \int_{-1}^{+1} I(\mu)\, \mu\, d\mu \qquad erg\ s^{-1}\, cm^{-2} \quad \text{-- (4.3)}$$

If isotropic, $I(\mu) = I$

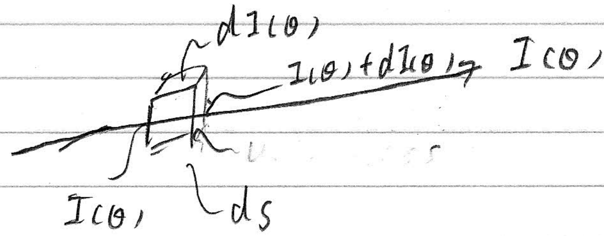
$$F = 2\pi I \int_{-1}^{+1} \mu\, d\mu = 2\pi I \cdot \frac{1}{2}\mu^2 \Big|_{-1}^{+1} = 0$$

no net radiation flux, $L = 0$

Therefore, $I(\mu)$ is not isotropic

\# Radiation Transfer Function

How $I(\theta)$ changes along its path s : $dI(\theta)$



* First, consider emission only

$$dI(\theta) = j\rho\, ds$$

j : mass emission coefficient : $erg\ s^{-1}\ g^{-1}$

$j\rho$: volume emission $erg\ s^{-1}\ cm^{-3}$

$j\rho ds$: intensity change $erg\ s^{-1}\ \underbrace{cm^{-3} \cdot cm}_{cm^{-2}}$

emissions are caused by ① spontaneous emission
② stimulated emission
③ free - free emission
④ scattering

* Second, consider absorption only

$$dI(\theta) = -\kappa\rho I(\theta)\, ds$$

absorption is proportional to incident radiation

* κ : ~~absor~~ opacity

or mass absorption coefficient $\underline{cm^2\ g^{-1}}$ absorption cross section

$\kappa\rho$: $cm^2\ g^{-1}\ g/cm^3 = cm^2 / cm^3$

absorption cross section unit volume

$\kappa\rho ds$: $cm^2/cm^3 \cdot cm$: dimensionless change rate

κ : opacity . caused by ① scattering
② various absorptions { free - free, bound - free, bound - bound

* Opacity κ (continued)

$$dI = -\kappa\rho I\,ds$$

$$\frac{dI}{I} = -\kappa\rho\,ds = d\ln I$$

$$\Rightarrow \ln I - \ln I_0 = -\kappa\rho s \Rightarrow \ln\frac{I}{I_0} = -\kappa\rho s$$

$$\Rightarrow I = I_0 e^{-\kappa\rho s}$$

$$\Rightarrow I = I_0 e^{-\frac{s}{\lambda}}$$

$\lambda = \frac{1}{\kappa\rho}$: the e-folding distance of radiation intensity

equivalent to mean free path of a photon

$$\lambda = \frac{1}{\underbrace{\kappa}_{\text{mass absorption cross section per unit mass}}\rho} = \frac{1}{\underbrace{\sigma}_{\text{absorption cross section of a single particle (microscopic)}} n}$$

σn : total cross section per unit volume

$\kappa\rho$: total cross section unit volume

$$\kappa = \sigma\frac{n}{\rho} = \sigma\frac{N_A}{\mu}$$

σ : is known from microphysics

$\boxed{\text{The Equation}}$: total change across ds

$$dI(\theta) = j\rho\,ds - \kappa\rho I(\theta)\,ds$$

$$\boxed{\frac{1}{\rho}\frac{dI(\theta)}{ds} = j - \kappa I(\theta)} \quad (4\text{-}4)$$

Eq. of Radiative Transfer

If isotropic and uniform, e.g. in blackbody

$$\frac{dI(\theta)}{ds} = 0 \Rightarrow I = \frac{j}{\kappa} = B(T) = \frac{\sigma T^4}{\pi}$$

Standard form of Radiative Transfer Equation

Introduce optical depth τ.

τ : is defined along reference axis, i.e. $\vec{r}$ or $\vec{z}$
is dimensionless

$$d\tau = -\kappa\rho\, dz = -\frac{dz}{\lambda}$$

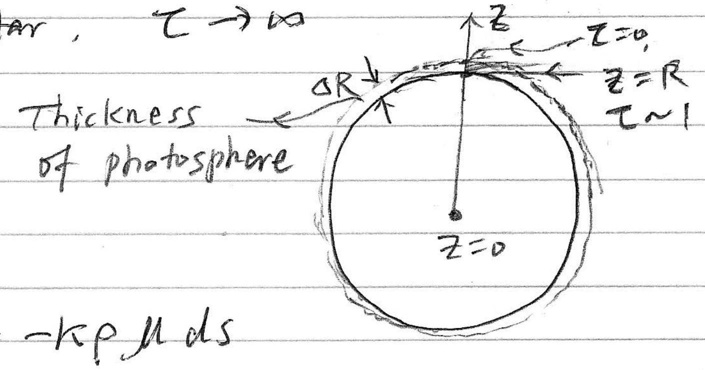
τ : dimensionless length normalized by photon mean free path

$\tau = 1$; $z = \lambda$. mean free path
the medium becomes opaque

$z = R$ True surface of star ; $\tau = 0$

$z = R$. visible surface of star
photospheric surface of star , $\tau \approx 1$, $\tau = \frac{2}{3}$

$z = 0$, center of star , $\tau \to \infty$



$$dz = ds \cos\theta$$

$$d\tau = -\kappa\rho \cos\theta\, ds = -\kappa\rho\mu\, ds$$

$$ds = -\frac{d\tau}{\kappa\rho\mu}$$

$$-\frac{1}{\rho}\frac{dI(\theta)}{\frac{d\tau}{\kappa\rho\mu}} = j - \kappa I(\theta)$$

$$\Rightarrow \mu \frac{dI(\mu)}{d\tau} = I(\mu) - \frac{j}{\kappa}$$

Introduce $\boxed{\text{source function } S = \frac{j}{\kappa}}$

$$\Rightarrow \mu \frac{dI(\mu)}{d\tau} = I(\mu) - S \quad \text{- - - - (4.6)}$$

(continued): The general form

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau_\nu} = I_\nu(\tau,\mu) - S_\nu(\tau,\mu) \quad \text{--- (4.10)}$$

when τ and S are a function of frequency
e.g. spectral lines

S is not isotropic $S(\mu)$

Solving the Transfer Equation

$$\mu \frac{dI}{d\tau} = I - S$$

This can be done analytically. First order normal differential Eq.

$$\frac{dI}{d\tau} - \frac{I}{\mu} = -\frac{S}{\mu} \quad \text{multiply by } e^{-\frac{\tau}{\mu}}$$

$$e^{-\frac{\tau}{\mu}}\frac{dI}{d\tau} - e^{-\frac{\tau}{\mu}}\frac{I}{\mu} = -e^{-\frac{\tau}{\mu}}\frac{S}{\mu}$$

$$\frac{d}{d\tau}\left(e^{-\frac{\tau}{\mu}} I\right) = -e^{-\frac{\tau}{\mu}}\frac{S}{\mu}$$

$$e^{-\frac{\tau}{\mu}} I(\tau)\Big|_{\tau_0}^{\tau} = -\int_{\tau_0}^{\tau} e^{-\frac{t}{\mu}}\frac{S(t)}{\mu}dt$$

τ_0: reference point $\quad \tau_0 = 0$ surface

$\tau_0 = \infty$ center

t: dummy integration variable

$$e^{-\frac{\tau}{\mu}} I(\tau) - e^{-\frac{\tau_0}{\mu}} I(\tau_0) = \int_{\tau}^{\tau_0} e^{-\frac{t}{\mu}}\frac{S(t)}{\mu}dt$$

multiply by $e^{\tau/\mu}$ on both sides

$$\Rightarrow \boxed{I(\tau,\mu) = e^{-(\tau_0-\tau)/\mu} I(\tau_0,\mu) + \int_{\tau}^{\tau_0} e^{-(t-\tau)/\mu}\frac{S(t)}{\mu}dt} \quad \text{--- (4.11)}$$

General solution

(continued)

Stellar interior. consider two different cases

① Forward directed radiation → toward surface

$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \mu \le 1$$

reference point $\tau_0 = \infty$, at the center

$$\underbrace{e^{-(\tau_0 - \tau)/\mu}}_{0} I(\tau_0, \mu) = 0$$

$$I(\tau, \mu > 0) = \int_\tau^\infty e^{-(t-\tau)/\mu} \cdot \frac{S(t)}{\mu} dt \quad \text{---- (4.12)}$$

② Inward - directed radiation → toward the center

$$\frac{\pi}{2} \le \theta \le \pi \quad -1 \le \mu \le 0, \text{ or } \mu \le 0$$

reference point $\tau_0 = 0$ true surface

$$\Rightarrow I(\tau_0, \mu \le 0) = 0$$

no incidence radiation from true surface

$$I(\tau, \mu \le 0) = \int_\tau^0 e^{-(t-\tau)/\mu} \frac{S(t)}{\mu} dt \quad \text{--- (4.13)}$$

To solve ① and ②, need to specify source function $S(t)$

Assuming source function to be near the integrated planck value $B(\tau)$ at depth τ

— A reasonable assumption, ① near LTE. ② not far from τ

$$S(t) = B(\tau) + (t-\tau)\left(\frac{\partial B}{\partial t}\right)_\tau$$

Expand $S(t)$ in a Taylor series, and keep the first order

$$① \quad I(\tau, \mu > 0) = \int_\tau^\infty e^{-(t-\tau)/\mu} \left[\frac{B(\tau)}{\mu} + \frac{(t-\tau)}{\mu}\left(\frac{\partial B}{\partial t}\right)_\tau\right] dt$$

$$I(\tau, \mu > 0) = -\mu \frac{B(\tau)}{\mu} e^{-(t-\tau)/\mu}\Big|_\tau^\infty + \mu \int_\tau^\infty \left(\frac{\partial B}{\partial t}\right)_\tau e^{-(t-\tau)/\mu} \left(\frac{t-\tau}{\mu}\right) d\frac{t-\tau}{\mu}$$

$$= (0 + B(\tau)) + \mu \left(\frac{\partial B}{\partial t}\right)_\tau \underbrace{\int_0^\infty e^{-x} x \, dx}_{=1}$$

(continued)

$$I(\tau, \mu \geq 0) = B(\tau) + \mu\left(\frac{\partial B}{\partial \tau}\right)_\tau \quad \text{--- (4.15)}$$

Similarly ②

$$I(\tau, \mu \leq 0) = B(\tau)\left[1 - e^{\tau/\mu}\right] + \mu\left(\frac{\partial B}{\partial \tau}\right)_\tau\left[e^{\tau/\mu}\left(\frac{\tau}{\mu} - 1\right) + 1\right] \quad \text{--- (4.16)}$$

$\mu \leq 0$ and $\underline{\tau \gg 1 \text{ in the interior}}$: $e^{\tau/\mu} = 0$

$$I(\tau, \mu \leq 0) = B(\tau) + \mu\left(\frac{\partial B}{\partial \tau}\right)_\tau \quad \text{--- (4.16)}$$

\# Radiation flux in the interior of a star : $\tau > 1$

$$F(\tau) = 2\pi \int_{-1}^{+1} I(\mu)\, \mu\, d\mu$$

$$= 2\pi \int_{-1}^{+1} \left(B(\tau) + \mu \frac{\partial B}{\partial \tau}\right) \mu\, d\mu$$

$$= 2\pi\left(\int_{-1}^{+1} B(\tau) \underbrace{\mu\, d\mu}_{\frac{1}{2}\mu^2\big|_{-1}^{1} = 0} + \int_{-1}^{+1} \frac{\partial B}{\partial \tau} \underbrace{\mu^2\, d\mu}_{\frac{1}{3}\mu^3\big|_{-1}^{+1} = \frac{2}{3}}\right)$$

$$\boxed{F(\tau) = \frac{4\pi}{3}\frac{\partial B(\tau)}{\partial \tau}} \quad \text{--- (4.17)}$$

net radiation flux

$$L(r) = 4\pi r^2 F(r) \quad \text{--- (4.18)}$$