CH3. Equation of state (Continued)

Review

* Distribution function

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp\left[\frac{-\mu + \varepsilon_j + \varepsilon(p)}{kT}\right] \pm 1}$$

+ : Fermion $s = \frac{1}{2}, \frac{3}{2}$

− : Boson $s = 0, 1$

$n = \int_p n(p)\, 4\pi p^2\, dp$ $\quad$ cm^{-3}

$\rho = \frac{n}{N_A}\mu$ $\quad$ g cm^{-3}

or $n = \frac{\rho}{\mu} N_A$

$P = \frac{1}{3}\int_p p v\, n(p)\, 4\pi p^2\, dp$ $\quad$ dyns cm^{-2}

$E = \int_p \varepsilon(p)\, 4\pi p^2\, dp$ $\quad$ ergs cm^{-3}

$P = (\gamma - 1) E$: "γ-Law" of Equation of state

① Blackbody radiation : photons in LTE, (T)

$n = 20.98\, T^3$ $\quad$ cm^{-3}

$\rho = 0$

$P = \frac{1}{3} a T^4$ $\quad$ dyns cm^{-2}

$E = a T^4$ $\quad$ erg cm^{-3}

$\gamma = \frac{4}{3}$ $\quad$ $\langle \varepsilon \rangle = 2.8\, kT$

$n(p) = n(\nu)$ = Planck Distribution

② Ideal Monatomic gas : non-interacting gas particles

$$n = \frac{g(2\pi m k T)^{3/2}}{h^3} e^{\mu/kT} e^{-\varepsilon_0/kT} \quad \text{cm}^{-3}$$

$P = nkT$

$E = \frac{3}{2} nkT$ $\quad$ $\langle \varepsilon \rangle = \frac{3}{2} kT$

$\gamma = \frac{5}{3}$

$n(p)$: Maxwell − Boltzmann Distribution

(3) Saha Equation — Ionization Rate

chemical reaction $\sum_i \mu_i = 0$

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_H/kT}$$

$y = \frac{n^+}{n}$: ionization fraction

$\Rightarrow$ thermal instability

(4) Boltzmann Population Distribution:

— Different energy states of an atom

$\Rightarrow$ strength of spectral lines

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left[-\frac{(\varepsilon_1-\varepsilon_2)}{kT}\right]$$

Now. # Degenerate gas: n, ρ, P, E, γ

degenerate electrons in White Dwarfs

degenerate neutrons in neutron stars

non-ideal gas: electrons mutually exclusive

$\langle \varepsilon \rangle \gg kT$ due to the quantum effect

Mixture of Ideal gas and Radiation

e.g. high mass stars, giant stars

Mixture of Ideal gas and Degenerate gas

e.g. Later stages of stellar equation. when nuclear fuel in the core exhausted.

Other thermodynamic derivatives:

χ_ρ, χ_T : power-law expression of $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$

$\Gamma_1, \Gamma_2, \Gamma_3$ and γ : adiabatic exponents.

Fermi - Dirac Equation of state

Fermions : $s = \frac{1}{2}$, electron, proton, neutron

$g = 2$,

$\varepsilon(p) >> kT$

$\varepsilon_j = \varepsilon_0 = mc^2$. relativistic term

$$n(p) = \frac{1}{h^3} \frac{2}{\exp\left[\frac{-\mu + mc^2 + \varepsilon(p)}{kT}\right] + 1}$$

$$n = \int_0^\infty n(p)\, 4\pi p^2 dp = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp\left[\frac{-\mu + mc^2 + \varepsilon(p)}{kT}\right] + 1} \quad \text{--- (3.44)}$$

$$\varepsilon(p) = (p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2 \quad (3.11)$$

$$\varepsilon(p) = mc^2 \left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right] \quad \text{--- (3.45)}$$

$$v(p) = \frac{\partial \varepsilon(p)}{\partial p} = \frac{p}{m}\left[1 + \left(\frac{p}{mc}\right)^2\right]^{-\frac{1}{2}} \quad \text{--- (3.46)}$$

Complete Degenerate Gas

- Based on the assumption that temperature is almost zero $T \to 0$ effectively.

* Fermi energy $\varepsilon_F = \mu - mc^2$
is a critical energy

$\varepsilon > \varepsilon_F$, $\quad \varepsilon - \mu + mc^2 = \varepsilon - \varepsilon_F > 0$

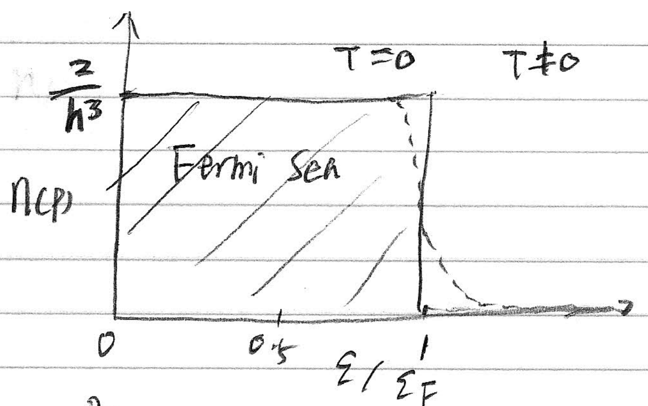
$$e^{\frac{\varepsilon - \varepsilon_F}{kT}} \to \infty \quad \Rightarrow n(p) = 0$$

$\varepsilon < \varepsilon_F$ $\quad \varepsilon - \varepsilon_F < 0$

$$e^{\frac{\varepsilon - \varepsilon_F}{kT}} = 0$$

$$\Rightarrow n(p) = \frac{2}{h^3} \doteq \text{a constant}$$

Fermions are only contained in the range $0 \leq \varepsilon \leq \varepsilon_F$
"Fermi sea"



$$n(p) = \int_0^{p_F} \frac{2}{h^3} 4\pi p^2 dp \qquad p_F: \text{ Fermi momentum}$$

$$x_F = \frac{p_F}{mc} \quad : \text{ dimensionless Fermi momentum}$$

$$x = \frac{p}{mc}$$

$$n = 8\pi \left(\frac{h}{mc}\right)^{-3} \int_0^{x_F} x^2 dx = \frac{8\pi}{3} \left(\frac{h}{mc}\right)^{-3} x_F^{\,3} \quad \text{---} (3.49)$$

For electrons, $m = m_e$, $\lambda_e = \frac{h}{mc}$: electron Compton wavelength

$$n_e = 5.865 \times 10^{29} x_F^3 \ cm^{-3} \quad \text{---} (3.50)$$

What is ρ ?

$$\rho = \frac{n_e}{N_A} \mu_e$$

$$\frac{\rho}{\mu_e} = \frac{n_e}{N_A} = B x_F^{\,3} \quad \text{---} (3.51)$$

and $B = \frac{8\pi}{3 N_A} \left(\frac{h}{m_e c}\right)^{-3} = 9.739 \times 10^5 \ g \ cm^{-3}$ --- (3.52)

For typical white dwarf, $x_F \approx 1$

He dwarf star $\mu_e = \left[\frac{X_i Z_i Y_i}{A_i}\right]^{-1} = \left[\frac{1 \cdot 2 \cdot 1}{4}\right]^{-1} = 2$

$\rho \simeq 2 \cdot B \cdot x_F^{\,3} \simeq 2 \times 10^6 \ g \ cm^{-3}$, or 2 tons / cm^3

For neutron stars, $m = m_p$, $\varepsilon_F = 0.3$

$\rho \simeq 2.7 \times 10^{14} \ g \ cm^{-3}$

In typical white dwarf,

$x_F \approx 1$, electrons move at a speed close to "c" 3×10^5 km/s

For a temperature of 10^8 K, what is the thermal speed?

$$\langle \varepsilon \rangle = \frac{1}{2} m v^2 = \frac{3}{2} KT$$

$$v = \sqrt{\frac{3KT}{m_e}} = \sqrt{\frac{3 \cdot 1.38 \times 10^{-16} \cdot 10^8}{9.10 \times 10^{-28}}} = 6.7 \times 10^9 \text{ cm/s}$$

$v \approx 7 \times 10^4$ km/s , $v < c$

$$\frac{\varepsilon(p)}{KT} \sim 50 \quad \Rightarrow \quad \varepsilon(p) \gg KT$$

$$P_e = \frac{8\pi}{3} \frac{m_e^4 c^5}{h^3} \int_0^{x_F} \frac{x^4 dx}{(1+x^2)^{\frac{1}{2}}} = A f(x_F) \quad \text{--- (3.53)}$$

$$A = \frac{\pi}{3} \left(\frac{h}{m_e c}\right)^{-3} m_e c^2 = 6.002 \times 10^{22} \text{ dyne cm}^{-2} \quad \text{--- (3.54)}$$

In constrast, constant density model of ideal gas

$$P_c = 1.34 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{ dyne cm}^{-2}$$

$R/R_\odot = 10^{-2}$, $\quad P_c = 10^7$ dyne cm^{-3}

P_{gas} $(10^7) \ll P_e$ (10^{22})

$$f(x) = x(2x^2 - 3)(1+x^2)^{\frac{1}{2}} + 3 \sinh^{-1} x \quad \text{--- (3.55)}$$

Similarly,

$$E_e = A g(x)$$

$$g(x) = 8x^3 [(1+x^2)^{\frac{1}{2}} - 1] - f(x) \quad \text{--- (3.57)}$$

$\Rightarrow$ $x \ll 1$, non-relativistic limit

$P_e \propto E_e \propto \left(\frac{\rho}{\mu_e}\right)^{5/3}$, and $\gamma = \frac{5}{3}$, like "ideal gas"

$\Rightarrow$ $x \gg 1$, extreme-relativistic limit

$P_e \propto E_e \propto (\rho/\mu_e)^{4/3}$, and $\gamma = 4/3$, like "photons"

(6)

Note : metal degenerates in room temperature

$$n_e = 5.865 \times 10^{29}\, x_F^{\,3} \quad \text{--} \quad cm^{-3} \quad \text{---} (3.50)$$

Cu metal : copper $Z = 29$, $A = 63.5$ amu

At room temperature $T = 300$ K

$$\rho = 9\ g/cm^3$$

$$n_e \sim 9 \times 10^{22}\ cm^{-3}$$

$$x_F \sim 6 \times 10^{-3} \quad (\text{non-relativistic})$$

$$\varepsilon_F = 7.1\ eV$$

$$V_F = 1.6 \times 10^8\ cm/s = 1600\ km/s$$

However, $V_{Th} \approx 10\ km/s$. electron velocity due to thermal motion at room temperature

$$V_F \gg V_{th}$$

Degenerate pressure dominates.

\# Application to White Dwarfs (3.5.2)

→ Radius – Mass Relationship

Stars of ideal gas, e.g. main sequence star

$$\frac{R}{R_\odot} = \left(\frac{M}{M_\odot}\right)^{0.75} \quad \text{---} \quad (1.87)$$

obtained from dimensional analysis

$M\uparrow, \quad R\uparrow$

White dwarf: — non-relativistic limit $x_F \ll 1$

$$E_e = A\,\underline{g(x_F)} = 6\times10^{22}\cdot \underline{\frac{12}{5}x^5} \quad \text{erg cm}^{-3}$$

$$\frac{\rho}{\mu_e} = Bx^3 = 9.74\times10^5\,x^3 \quad \text{g cm}^{-3}$$

Dimensional analysis

$$E_e \propto \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Total Internal Energy:

$$U = E_e V = E_e\cdot\frac{4}{3}\pi R^3 \propto \left(\frac{\rho}{\mu_e}\right)^{5/3} R^3$$

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \quad : \text{assume constant density}$$

$$U \propto \frac{M^{5/3}}{\mu_e^{3/5}}\frac{1}{R^2}$$

Apply Virial Theorem: $3(\gamma-1)U = -\Omega = q\dfrac{GM^2}{R}$

$$\frac{M^{5/3}}{\mu_e^{5/3}}\frac{1}{R^2} \propto \frac{M^2}{R}$$

$$M \propto \frac{1}{\mu_e^5}\frac{1}{R}$$

Or exactly: $\dfrac{M}{M_\odot} = 10^{-6}\left(\dfrac{R}{R_\odot}\right)^{-3}\left(\dfrac{2}{\mu_e}\right)^5 \quad \text{---} \quad (3.63)$

$M\uparrow, \quad R\downarrow$; typically $\mu_e = 2$

e.g. $M = 0.6\,M_\odot \Rightarrow R \approx 0.01\,R_\odot = R_{Earth}$

(continued)

Similarly, for neutron stars

$$\frac{M}{M_\odot} = 5 \times 10^{-15} \left(\frac{R}{R_\odot}\right)^{-3} \quad \text{---} \quad (3.64)$$

For $M = M_\odot$, $R = 11$ km

Consider the limit of extremely relativistic case

$x_F >> 1$

$$E_e \propto \left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}}$$

$$U \propto \frac{M^{\frac{4}{3}}}{\mu_e^{3/4}} \frac{1}{R^4} R^3 \propto \frac{M^2}{R}$$

$$\Omega \propto \frac{M^2}{R}$$

$$\frac{M}{M_\odot} = 1.456 \left(\frac{2}{\mu_e}\right)^2 \quad \text{------} \quad (3.67)$$

M is a constant

$\boxed{M = 1.456\, M_\odot \quad \text{Chandrasekhar limit mass}}$

The limit of White Dwarf stars

3.5.3 Effects of Temperature on Degeneracy

Transition from non-degeneracy to degeneracy state

$$\underbrace{\varepsilon_F}_{\text{Fermi energy}} = \underbrace{kT}_{\text{thermal energy}}$$

$$\Rightarrow \frac{\rho_F}{\mu_e} \simeq 6.0 \times 10^{-9}\, T^{3/2} \text{ g cm}^{-3}$$

e.g. Sun center $T = 15 \times 10^6$ K, $\mu_e = 1.2$, $\rho_c = 80$ g cm^{-3}

$\rho_F = 412$ g cm^{-3}, $\rho_F > \rho_c$ $\boxed{\text{non-degenerate}}$

3.7 Thermodynamic Derivatives, adiabatic exponents

known: T, n, ρ, P, E

* Specific Heats

First Law of Thermal Dynamics

$$dQ = dE + P\,dv_\rho = dE + P\,d\left(\frac{1}{\rho}\right) = dE - \frac{P}{\rho^2}\,d\rho$$

specific volume: cm³/g

$$C_{v_\rho} = \left(\frac{dQ}{dT}\right)_{v_\rho} = \left(\frac{\partial E}{\partial T}\right)_\rho \quad erg\ g^{-1}\ K^{-1}$$

specific heat of constant density or volume deg

= energy needed to raise one gram of gas by one K

For ideal gas: $E = \frac{3}{2} nkT = \frac{3}{2}\frac{\rho}{\mu} N_A kT \quad erg\ cm^{-3}$

$$E = \frac{E}{\rho} = \frac{3}{2}\frac{N_A}{\mu} kT \qquad erg\ g^{-1}$$

$$C_v = \left(\frac{\partial E}{\partial T}\right)_\rho = \frac{3}{2}\frac{N_A}{\mu} k$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial\left(E + \frac{P}{\rho}\right)}{\partial T}\right)_p$$

$$dQ = dE + p\,dv = dE + d\underbrace{pv}_{P/\rho} - v\,dp = d\left(E + \frac{P}{\rho}\right) - \frac{1}{\rho}\,dp$$

* Power-Law expression of the equation of state

$$P = P_0 \rho^{\chi_\rho} T^{\chi_T}$$

Ideal gas: $\chi_\rho = 1$, $\chi_T = 1$

photons: $\chi_\rho = 0$, $\chi_T = 4$

$$\ln P = \ln P_0 + \chi_\rho \ln \rho + \chi_T \ln T$$

$$d\ln P = \chi_\rho\, d\ln\rho + \chi_T\, d\ln T$$

$$\Rightarrow \chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T \ ; \quad \chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho \quad \text{--} \quad \begin{matrix}(3.88)\\(3.89)\end{matrix}$$

χ_ρ, χ_T: dimensionless parameters

$$d\ln P = \frac{dP}{P}$$

Scale height: H : $P = P_0 e^{-\frac{r}{H}}$.

H: pressure change by a factor of e over H.

H small, high gradient

H large, small gradient

$$d\ln P = \frac{dP}{P} = \frac{P_0\, d(e^{-\frac{r}{H}})}{P_0 e^{-\frac{r}{H}}} = \frac{\cancel{P_0 e^{-\frac{r}{H}}}\cdot(-\frac{1}{H})dr}{\cancel{P_0 e^{-\frac{r}{H}}}} = -\frac{dr}{H}$$

Therefore $\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = \left(\frac{-\frac{dr}{H_P}}{-\frac{dr}{H_T}}\right)_\rho = \left(\frac{H_T}{H_P}\right)_\rho$

χ_T: ratio of temperature scale height H_T over pressure scale height H_P

* γ : another γ

$\gamma = \frac{C_P}{C_V}$: ratio of specific heats

$$\gamma = 1 + \frac{P}{\rho T C_{V_\rho}} \frac{\chi_T^2}{\chi_\rho} \quad \text{---} \quad (3.92)$$

* Adiabatic Exponents : Γ_1, Γ_2, Γ_3

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{ad} = \left(\frac{H_\rho}{H_P}\right)_{ad} \quad \text{---} \quad (3.93)$$

$P = P_0 \rho^{\Gamma_1}$ if adiabatic

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} = \left(\frac{H_T}{H_P}\right)_{ad} = \frac{1}{\nabla_{ad}} \quad \text{---} \quad (3.94)$$

∇_{ad}: dimensionless temperature gradient (chap. 5). criterion for convection

$P = P_0 T^{\frac{\Gamma_2}{\Gamma_2 - 1}}$, if adiabatic

(continued)

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{ad} = \left(\frac{H_\rho}{H_T}\right)_{ad}$$

$$T = T_0 \rho^{\Gamma_3 - 1} \quad \text{if adiabatic}$$

How to calculate $\Gamma_1, \Gamma_2, \Gamma_3$ for given states

$$\Gamma_3 - 1 = \frac{P}{\rho T}\frac{\chi_T}{C_{V\rho}} = \frac{1}{\rho}\left(\frac{\partial P}{\partial E}\right)_\rho$$

because $\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = \frac{\frac{1}{P}\partial P}{\frac{1}{T}\partial T}$, and $\partial E = C_{V\rho}\,\partial T$

$$\Rightarrow P = (\Gamma_3 - 1)\rho E$$

Recall "γ-law" of equation of state.

$\boxed{\text{this } \gamma \text{ is } \Gamma_3}$ $\quad \Gamma_3 - 1 = \frac{P}{\rho T}\frac{\chi_T}{C_{V\rho}}$

$$\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\rho$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \nabla_{ad}^{-1} = \frac{\chi_\rho}{\Gamma_3 - 1} + \chi_T$$

$$\gamma = \frac{C_P}{C_V} = \frac{\Gamma_1}{\chi_\rho} = 1 + \frac{\chi_T}{\chi_\rho}(\Gamma_3 - 1)$$

Ideal gas: $\chi_\rho = \chi_T = 1$

$$\Gamma_3 = \Gamma_2 = \Gamma_1 = \frac{5}{3}, \quad \gamma = \frac{5}{3}$$

Radiation: $\chi_\rho = 0, \quad \chi_T = 4$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \frac{4}{3}, \quad \gamma = \frac{\Gamma_1}{\chi_\rho} = \infty$$

Mixture of Ideal Gas and Radiation

$$P_g = nkT = \frac{\rho}{\mu} N_A kT$$

$$P_{rad} = \frac{1}{3} a T^4$$

Total $P = P_g + P_{rad} = P_0 \rho^{\chi_\rho} T^{\chi_T}$

Introduce β: $\beta = \frac{P_{gas}}{P}$, ratio of gas pressure to total pressure

$\Rightarrow$ χ_ρ, χ_T in terms of β?

Looking for $d\ln P = \chi_\rho \, d\ln\rho + \chi_T \, d\ln T$

$$P = P_g + P_{rad}$$

$$\frac{dP}{P} = \frac{dP_g}{P} + \frac{dP_{rad}}{P}$$

$$P = P_g/\beta \quad \leftarrow \quad \frac{P_g}{P} = \beta$$

$$P = P_{rad}/(1-\beta) \quad \leftarrow \quad \frac{P_{rad}}{P} = 1-\beta$$

$$\frac{dP}{P} = \beta \frac{dP_g}{P_g} + (1-\beta) \frac{dP_{rad}}{P_{rad}}$$

$$d\ln P = \beta \, d\ln P_g + (1-\beta) \, d\ln P_{rad}$$

From ideal gas $\quad d\ln P_g = d\ln\rho + d\ln T$

photon $\quad d\ln P_{rad} = 4 \, d\ln T$

$$d\ln P = \beta \, d\ln\rho + \underbrace{(\beta + 4(1-\beta))}_{4-3\beta} d\ln T$$

$$\Rightarrow \chi_\rho = \beta$$

$$\chi_T = 4 - 3\beta$$

Ideal gas. $\beta = 1$, $\chi_\rho = 1$, $\chi_T = 1$

photon $\beta = 0$, $\chi_\rho = 0$, $\chi_T = 4$

continued.

$$C_{V_p} = \frac{3 N_A k}{2\mu}\left(\frac{8-7\beta}{\beta}\right) \text{ erg } g^{-1} K^{-1}$$

$$\Gamma_3 - 1 = \frac{2}{3}\left(\frac{4-3\beta}{8-7\beta}\right)$$

$$\Gamma_1 = \beta + (4-3\beta)(\Gamma_3 - 1)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{32-24\beta-3\beta^2}{2(4-3\beta)}$$

$$\gamma = \frac{\Gamma_1}{\beta}$$