CH3: Equation of state

state: the state of the union of particles, including gas particles, e.g., H, He, e^- and photons; γ.

state parameters: statistical properties of the union

T: temperature

ρ: (mass) density $\longleftrightarrow$ $n = \frac{\rho}{\mu} N_A$ number density

P: pressure

E: internal energy

* why matters?

In hydrostatic equilibrium equation

$$\frac{dP}{dr} = -\rho g$$

Essential to know P: pressure

$P = P(\rho, T)$; for any given ρ, T.

For an ideal gas

$$P = nkT = \frac{\rho}{\mu} N_A \cdot kT \quad ; \quad P = P_{gas}$$

μ: a function of composition & ionization

However, stellar interior is not ideal

1. become degenerate when density gets high
2. radiation pressure $P_{rad} \uparrow$, when temperature high
3. relativistic effect when particle velocity high

* What matters?

$P = P(\rho, T)$.

P_{gas} : gas pressure $= nkT$

P_{rad} : radiation pressure $= \frac{1}{3} a T^4$

P_e : electron degenerate pressure $= \text{const.} \rho^{5/3}$

Distribution function (CH3.1)

— From statistical mechanics

Also called partition function

Also called occupation number.

General form: particle number in coordinate - momentum space

$$f(\vec{x}, \vec{p}, t) = f(x, y, z, P_x, P_y, P_z, t)$$

hydrostatic : drop "t"

local homogeneous : drop "x," "y," "z"

isotropic : $P_x, P_y, P_z \rightarrow P$: momentum sphere $4\pi P^2 dP$

Seven - dimension $\longrightarrow$ One dimension of P

$$\boxed{n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\underbrace{\exp\{[-\mu + \varepsilon_j + \varepsilon(p)]/kT\}}_{\text{classical term}} \underbrace{\pm 1}_{\text{quantum term}}}} \quad \text{--- (3.9)}$$

* h : planck's constant 6.626×10^{-23} erg s ($\Delta E \cdot \Delta t$), or ($\Delta x \cdot \Delta p$)

* $n(p)$: number of particles per unit volume per unit momentum

number density $\left\{ \boxed{n = \int_p n(p) \underbrace{4\pi p^2 dP}_{\text{momentum sphere}} \quad cm^{-3}} \right.$ -- (3.10)

Integration over momentum space

* P : momentum : $\begin{cases} p = mv & \text{non-relativistic} \\ p = \gamma m v & \text{relativistic} \end{cases}$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$. Lorentz factor

* $\varepsilon(p)$: particle kinetic energy $\begin{cases} \frac{p^2}{2m} = \frac{1}{2} m v^2 & \text{non-relativistic} \\ (p^2c^2 + m^2c^4)^{\frac{1}{2}} - mc^2 . & \text{relativistic} \end{cases}$

m : rest mass

mc^2 : rest mass energy

total particle energy

$$\varepsilon = \varepsilon(p) + mc^2 = \gamma m c^2$$
$$= (p^2c^2 + m^2c^4)^{\frac{1}{2}}$$

* T : temperature; assuming Local Thermal Equilibrium
LTE

LTE : locally, particles interact frequently enough, that establish a velocity distribution uniquely determined by a single parameter : T

e.g. $\langle \frac{1}{2} m v^2 \rangle = \frac{3}{2} kT$
of ~~Maxwellian~~ — Boltzmann distribution of gas particles

$T_{ion} = T_{electron} = T_{photon}$

e.g. of Planck function of photons
— Black Body radiation

* ε_j : energy state j of the particle
e.g., atomic energy level j.

In Hydrogen Bohr's model : j = 0 ground state
This is a relative energy, need to define $\boxed{\text{reference energy level}}$

H in the ground state : $\varepsilon_j = -13.6$ eV
reference energy : $\doteq 0$ for just ionized

e.g. for relativistic particles
$\varepsilon_j = mc^2$ rest mass energy

* g_j : number of degeneracy of state j, having same ε_j
e.g. number of spins,

* μ : chemical potential of particles

* k : $\doteq 1.38 \times 10^{-16}$ erg K^{-1} Boltzmann Constant

* "±1" term. Quantum effect.
when density is high, or temperature is low

"+" : for Fermion particles : e : electron
p : proton
n : neutron
having half-interger spin : $S = \frac{1}{2}$

"−" for Boson particles : γ : photon
having zero or integer spin . $S = 0$
or $S = 1$

* Deriving state parameters

$$\underbrace{P}_{\text{isotropic}} = \frac{1}{3} \int_p n(p) \underbrace{\mathbf{p} v}_{\text{momentum flux}} \underbrace{4\pi p^2 dp}_{\text{momentum sphere}} \qquad \text{--- } 3.13$$

$$v = \frac{\partial \varepsilon(p)}{\partial p} \qquad \text{--- } 3.12$$

$$E = \int_p n(p) \underbrace{\varepsilon(p)}_{\text{kinetic energy}} \underbrace{4\pi p^2 dp}_{\text{momentum sphere}} \qquad \text{--- } (3.14)$$

Blackbody Radiation (Ch 3.2)

photon: massless boson of unit spin (rest)

$$\exp\left[\frac{-\mu + \varepsilon_j + \varepsilon(p)}{kT}\right] = \exp\left[\frac{pc}{kT}\right]$$

One energy state: $\varepsilon_j = \varepsilon_0 = \underset{=0}{m_0} c^2 = 0$

$\mu = 0$

$$\varepsilon(p) = (p^2c^2 + \cancel{m^2c^4})^{\frac{1}{2}} - \cancel{mc^2} = pc$$

$g_i = 2$, spin-up & down

$$* \quad \boxed{n(p) = \frac{1}{h^3}\frac{2}{e^{\frac{pc}{kT}} - 1}}$$ distribution function for photons/radiation

* number density

$$n_\gamma = \int n(p)\, 4\pi p^2 dp = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{\exp\left[\frac{pc}{kT}\right] - 1} \quad cm^{-3}$$

Let $x = \frac{pc}{kT} = \frac{\varepsilon(p)}{kT}$

$$n_\gamma = \frac{8\pi k^3 T^3}{h^3 c^3}\underbrace{\int_0^\infty \frac{x^2 dx}{e^x - 1}}_{} = 2\zeta(3) = 2\cdot(1.202)$$

$$\boxed{n_\gamma = 2\pi\zeta(3)\left(\frac{2kT}{ch}\right)^3 \approx 20.28\, T^3} \quad cm^{-3}$$

* mass density

$\rho_\gamma = \frac{n_\gamma}{N_A}\mu = 0$, since $\mu = 0$, molecular weight $= 0$ for photons

* radiation pressure

$$P_{rad} = \frac{1}{3}\int_0^\infty n(p)\, pc\, 4\pi p^2 dp$$

$$P_{rad} = \frac{1}{3}\frac{8\pi k^4 T^4}{h^3 c^3}\int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$\boxed{P_{rad} = \left(\frac{k^4}{c^3h^3}\frac{8\pi^5}{15}\right)\frac{T^4}{3} = \frac{a}{3}T^4} \quad \text{dyne cm}^{-2} \quad \text{— (3.17)}$$

$$a = \frac{k^4}{c^3h^3}\frac{8\pi^5}{15} \quad : \quad \text{erg cm}^{-3}\,K^{-4}$$

$$= 7.56\times10^{-15} \text{ erg cm}^{-3}\,K^{-4}$$

— radiation constant.

$$E_{rad} = \int_p n(p)\frac{\varepsilon(p)}{pc}4\pi p^2 dp$$

$$\boxed{E_{rad} = aT^4} \quad \text{dyne cm}^{-2} \quad \text{——— (3.18)}$$

$$E_{rad} = 3P_{rad}$$

Using the general "γ-law" of equation of state

$$p = (\gamma-1)E = \frac{1}{3}E$$

$$\gamma - 1 = \frac{1}{3}, \quad \boxed{\gamma = \frac{4}{3}} \quad \text{for radiation}$$

$$\gamma = \frac{n+2}{n} = \frac{4}{3} \Rightarrow \boxed{n=6}$$

photon in six-dimension space

Useful Integration Identity:

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} \quad : \text{Gaussian Integral}$$

$$\int_0^\infty \sqrt{x}\,e^{-x}dx = \frac{1}{2}\sqrt{\pi} \quad : \text{Gamma function}$$

$$\int_0^\infty \frac{x}{e^x-1}dx = \frac{\pi^2}{6} \quad : \text{Bernulli number}$$

$$\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$$

$$\int_0^\infty \frac{x^2}{e^x-1}dx = 2\zeta(3) = 2\cdot(1.202)$$

* Planck function : photon distribution in ν, or λ
u_ν, u_λ frequency wavelength

$$E = \int_0^p n(p)\, \varepsilon(p)\, 4\pi p^2 dp = \int_0^\infty u_\nu d\nu = \int_0^\infty u_\lambda d\lambda$$

$$\varepsilon(p) = pc = h\nu = h\frac{c}{\lambda}$$

$$p = \frac{h}{c}\nu \,, \quad p = \frac{h}{\lambda}$$

$$E = \int_0^\infty \frac{2}{h^3(e^{\frac{h\nu}{kT}}-1)} \cdot h\nu \cdot 4\pi \frac{h^2\nu^2}{c^2} \frac{h}{c} d\nu = \int_0^\infty u_\nu d\nu$$

$$\boxed{u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}}-1}} \quad \text{erg cm}^{-3}\,\text{Hz}^{-1} \quad \text{--- (3.19)}$$

Similarly

$$\boxed{u_\lambda = \frac{8\pi h c}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}}-1}} \quad \text{erg cm}^{-3}\,\text{cm}^{-1} \quad \text{--- (3.20)}$$

Black body radiation flux

$$\boxed{B_\nu = \frac{u_\nu \cdot c}{4\pi} = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}}-1}} \quad \text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{srad}^{-1} \quad \text{--- (3.21)}$$

Integrated black-body radiation

$$\boxed{B = \int_0^\infty B_\nu d\nu = \frac{ca}{4\pi} T^4 = \frac{\sigma T^4}{\pi}} \quad \text{erg cm}^{-2}\,\text{s}^{-1}\,\text{std}^{-1} \quad (3.22)$$

$$\sigma = 5.67 \times 10^{-5} \ \text{erg cm}^{-2}\,\text{K}^{-4}\,\text{s}^{-1}$$

Stefan - Boltzmann constant

* Stefan-Boltzmann's Law : $F = \sigma T^4$
From a surface, the std angle $\Omega = \pi$

* Peak Radiation $dB_\nu / d\nu = 0$
$\boxed{x = \frac{h\nu}{kT} = 2.82}$: Average photon energy $\approx$ particle energy

Ideal Monatomic Gas (CH 3-3)

One energy level $\varepsilon_j = \varepsilon_0$

e.g. H in ground state $\varepsilon_0 = -13.6$ eV

μ: chemical potential $\left| \frac{-\mu}{kT} \right| \gg 1$

or $\mu \ll 0$

or "chemical reaction" not allowed.

or T is low

$\Rightarrow e^{\frac{-\mu}{kT}} \gg 1$.

$\Rightarrow$ quantum term "1" ignored

$\Rightarrow$ only classical term remain $\quad \varepsilon(p) = \frac{p^2}{2m}$

$$\Rightarrow \boxed{n(p) = \frac{1}{h^3} \frac{g}{\exp\left[\frac{-\mu + \varepsilon_0 + p^2/2m}{kT}\right]}}$$

number density

$$n = \int n(p)\, 4\pi p^2 dp$$

$$n = \frac{4\pi}{h^3} g \int_0^\infty e^{\frac{\mu}{kT}} e^{-\frac{\varepsilon_0}{kT}} e^{-\frac{p^2}{2mkT}} p^2 dp \quad \text{--- 3.23}$$

$$\boxed{n = \frac{g(2\pi mkT)^{\frac{3}{2}}}{h^3} e^{\frac{\mu}{kT}} e^{-\varepsilon_0/kT}} \quad \text{----}$$

number density depends on T, m, μ, ε_0

$$\boxed{e^{\frac{\mu}{kT}} = n \frac{h^3}{g(2\pi mkT)^{3/2}} e^{\varepsilon_0/kT}} \quad \text{——— 3.24}$$

chemical potential is related with n

(continued)

$$n = \int dn_{(p)} = \int n_{(p)}\, 4\pi p^2 dp$$

$$\frac{dn_{(p)}}{n} = \frac{n_{(p)}\, 4\pi p^2 dp}{n} = \frac{\frac{g}{h^3} \cancel{e^{\frac{\mu}{kT}}} \cancel{e^{-\frac{\varepsilon_0}{kT}}} e^{-\frac{p^2}{2mkT}} p^2 dp}{\frac{g(2\pi m kT)^{3/2}}{h^3} \cancel{e^{\frac{\mu}{kT}}} \cancel{e^{-\varepsilon_0/kT}}}$$

$$\boxed{\frac{dn_{(p)}}{n} = \frac{4\pi}{(2\pi m kT)^{3/2}} e^{-\frac{p^2}{2mkT}} p^2 dp} \quad \text{(3.25)}$$

Maxwell - Boltzmann Distribution of ideal gas particles

Similarly $\frac{dn_{(\varepsilon)}}{n} = \frac{2}{\pi^{\frac{1}{2}}} \cdot \frac{1}{(kT)^{3/2}} e^{-\frac{\varepsilon}{kT}} \varepsilon^{\frac{1}{2}} d\varepsilon$ .. (3.26)

~~Max~~ Peak particle energy: $dn_{(\varepsilon)}/d\varepsilon = 0$

$$\Rightarrow \varepsilon = \frac{kT}{2}$$

Average particle energy $\langle \varepsilon \rangle = \frac{\int \varepsilon\, dn_{(\varepsilon)}}{\int dn_{(\varepsilon)}} = \frac{\int \varepsilon\, dn}{n}$

$$\Rightarrow \boxed{\langle \varepsilon \rangle = \frac{3}{2} kT}$$

$$P = \frac{1}{3} \int n_{(p)}\, pv \cdot 4\pi p^2 dp$$

$$P = g \frac{4\pi}{h^3} \frac{\pi^{\frac{1}{2}}}{8m} (2mkT)^{\frac{5}{2}} e^{\frac{\mu}{kT}} e^{-\varepsilon_0/kT}$$

$\boxed{P = nkT}$ dyne cm^{-2} (3.28)

Similarly $E = \int n_{(p)}\, \varepsilon_{(p)}\, 4\pi p^2 dp$

$\boxed{E = \frac{3}{2} nkT}$ (3.29)

$P = (\gamma - 1) E$ $\quad (\gamma - 1) = \frac{2}{3} \Rightarrow \boxed{\gamma = \frac{5}{3}}$

(Continued)

* For same particle, but different energy state

$$\varepsilon_1 \neq \varepsilon_2, \quad g_1 \neq g_2$$

$$\mu_1 = \mu_2$$

$$n_1 = \frac{g_1 (2\pi m k T)^{\frac{3}{2}}}{h^3} e^{\frac{\mu_1}{kT}} e^{-\frac{\varepsilon_1}{kT}}$$

$$n_2 = \frac{g_2 (2\pi m k T)^{\frac{3}{2}}}{h^3} e^{\frac{\mu_2}{kT}} e^{-\frac{\varepsilon_2}{kT}}$$

$$\boxed{\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{-(\varepsilon_1 - \varepsilon_2)/kT}} \quad --- (3\text{-}30)$$

⇒ Boltzmann population distribution

Different energy levels of atoms are populated or depopulated by photon absorption or emission

⇒ determine the strength of spectral lines

Saha Equation (CH3.4)

Given T and ρ, what is the ionization rate.

Consider the "ionization - recombination" process

or "chemical reaction"

$$H^+ + e^- \Longleftrightarrow H^0 + \chi_H$$

$\chi_H = 13.6$ ev . Ionization energy from the ground state of Hydrogen H

Ionization fraction: $y = \frac{n^+}{n} = \frac{n_e}{n}$ in pure hydrogen

$$n = n^+ + n^0$$

	ε_0,	g	μ
H^+ :	0	1	μ^+
e^- :	0	2	μ^-
H^0 :	$-\chi_H$ $(-13.6\,ev)$	2	μ^0

reference energy level is just at the ionization state

Again, from equation (3.24), for ideal gas:

$$n = \frac{g(2\pi m kT)^{3/2}}{h^3} e^{\frac{\mu}{kT}} e^{-\varepsilon_0/kT} \quad \text{--- (3.24)}$$

$$n_e = \frac{2(2\pi m_e kT)^{3/2}}{h^3} e^{\frac{\mu^-}{kT}} \quad \text{--- (3.32)}$$

$$n^+ = \frac{(2\pi m_p kT)^{3/2} e^{\frac{\mu^+}{kT}}}{h^3} \quad \text{--- (3.33)}$$

$$n^0 = \frac{2[2\pi(m_e+m_p)kT]^{3/2}}{h^3} e^{\mu_0/kT} e^{\chi_H/kT} \quad \text{--- (3.34)}$$

(Continued)

$$\frac{n^+ n_e}{n^0} = \frac{(2\pi kT)^{3/2}}{h^3} \left(\frac{m_e m_p}{m_e + m_p}\right)^{\frac{3}{2}} e^{\frac{(\mu^- + \mu^+ - \mu^0)}{kT}} e^{-\chi_H/kT}$$

Since $\frac{m_e m_p}{m_e + m_p} \approx \frac{m_e \cdot m_p}{m_p} = m_e$

$\mu^- + \mu^+ - \mu^0 = 0$

$1H^+ + 1e^- = 1H^0$

$1e^- + 1H^+ - 1H^0 = 0$

$\Rightarrow \mu^- + \mu^+ - \mu^0 = 0$ ⇐ required by chemical equilibrium

chemical equilibrium: $\mu_i = \left(\frac{\partial E}{\partial N_i}\right)_{S,V}$

$$dE = \underbrace{TdS}_{dQ} - PdV + \mu_i dN_i$$

$$\sum_i \mu_i dN_i = 0$$

$$\Rightarrow \frac{n^+ n_e}{n^0} = \left(\frac{2\pi m kT}{h^2}\right)^{3/2} e^{-\frac{\chi_H}{kT}}$$

$$\frac{y^2}{1-y} = \frac{1}{n}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_H/kT} \quad \text{--- (3.29)}$$

Pure hydrogen: $\frac{y^2}{1-y} = \frac{4.01\times10^{-9}}{\rho} T^{3/2} \exp[-1.578\times10^5/T]$ --- (3.40)

Because of the exponential component, the ionization ~~potential~~ fraction is very sensitive to temperature

Half-ionization $y = \frac{1}{2}$ at T: $\frac{\chi_H}{kT} = 10$

H: $\chi_H = 13.6$ ev $\quad T \sim 1.5\times10^4$ K

He^+: $\chi_H = 24.6$ ev $\quad T \sim 3\times10^4$ K

He^{++}: $\chi_H = 54.4$ ev $\quad T \sim 6\times10^4$ K

(continued)

pure hydrogen: $y = \frac{1}{2}$, $T = 1.6 \times 10^4$ K

$y = 0$, $T = 1.0 \times 10^4$ K

$y = 1$, $T = 2.6 \times 10^4$ K

Therefore, ionization happens in a narrow temperature zone.

In red giants, as atmosphere cools down due to expansion

→ recombination happens, forming neutrals

→ opacity increases because of neutrals

→ radiation pressure increases

→ push further → cool more → radiation pressure ↑

→ run away

On the other hand,

If atmosphere contracts,

→ temperature increases

→ ionization more

→ opacity decrease

→ radiation pressure decrease

→ contracts more → run away

⇒ thermal instability

causing pulsation in supergiants

leading to strong stellar wind

and to planetary nebula : shedding