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CHAP. 1: Preliminaries (continued)

* Estimate stellar Temperature
* Constant density model
  - $P(r)$ Pressure Distribution, P_c at center
  - $T(r)$ Temperature Distribution, T_c
  - Molecular Weights, μ versus abundance
* Evolutionary Lifetime on the Main Sequence.

Recap of the Virial Theorem

General: $\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \underbrace{\sum_i F_i \cdot r_i}_{\text{Virial}}$

Gravitational System: $\frac{1}{2}\frac{d^2 I}{dt^2} = 2\underline{K} + \Omega$

State parameters: $\frac{1}{2}\frac{d^2 I}{dt^2} = \int_M \frac{3P}{\rho} dM_r + \Omega$

Energetics: $\frac{1}{2}\frac{d^2 I}{dt^2} = 3(\gamma - 1) U + \Omega$, since $P = (\gamma - 1)\rho E$

Relations between W, U, Ω

$$W = U + \Omega$$

$$W = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega$$

$$U = W - \Omega = \frac{-1}{3(\gamma-1)}\Omega$$

$\gamma = \frac{5}{3}$, monatomic Ideal gas

$$U = -\frac{1}{2}\Omega$$

$$W = \frac{1}{2}\Omega$$

When a star contracts, half gravitational energy goes to internal energy, the other half emits light

Estimate Stellar Temperature (CH 1.3.4)

- An application of Virial theorem
- Assume (1) uniform density, (2) uniform temperature

But, both assumptions are wrong

Thus, a ~~zeroth~~-order estimation.

$$U = -\frac{1}{2}\Omega \quad ,$$

$$\Omega = -\frac{3}{5}\frac{GM^2}{R} \quad \leftarrow \text{uniform density} ;$$

For monatomic ideal gas,

$$E = \frac{3}{2} n kT \quad \text{erg cm}^{-3} \quad \text{--- (1.34)}$$

Internal energy density

(specific $E \Rightarrow \rho E$ = erg g^{-1})

$$U = EV \; ; \quad V\text{: total volume.}$$

(Note: $P = (\gamma - 1)\rho E$: general equation of state)

$$\Rightarrow P = \frac{2}{3}\rho E = \frac{2}{3}\rho \cdot \frac{3}{2} n kT$$

$$\Rightarrow P = nkT \text{ . the familiar gas law)}$$

(Details of how to define E, P can be found in CHAP. 3)

$$\boxed{n = \frac{\rho}{\mu} N_A}$$

n: number density : particles / cm^3

ρ : mass density : gram / cm^3

μ: mean molecular weight in unit of amu (atomic mass unit)

1 amu = 1.66×10^{-24} gram

N_A : Avogadro's constant = 6.022×10^{23} mole^{-1}

$$\Rightarrow E = \frac{3}{2}\frac{N_A \cdot kT}{\mu} \cdot \rho$$

$$\frac{3}{2} \cdot \frac{N_A \cdot kT}{\mu} \cdot \rho \cdot V = +\frac{3}{10}\frac{GM^2}{R}$$

$$\Rightarrow T = 4.09 \times 10^6 \mu \left(\frac{M}{M_\odot}\right)^{2/3} \rho^{\frac{1}{3}} \; K \quad \text{---- (1.36)}$$

For the Sun, $\langle \rho \rangle = 1.4 \ g \ cm^{-3}$

$\Rightarrow T \sim 4 \ MK$

Therefore, hydrogen particles have to do nuclear fusion at this temperature

$\Rightarrow$ star is powered by nuclear energy

Detailed numerical calculation, $T_c = 15 \ MK$

$\rho_c = 80 \ g \ cm^{-3}$

Constant Density Model (CH 1-4)

Assume constant density, $\rho_{(r)} = \rho_c$ = constant

find $P_{(r)}$, P_c

$T_{(r)}$, T_c

First order of approximation. since $\rho_{(r)} = \rho_c$ is not valid

From the Hydrostatic Equilibrium Equation

$$\frac{dP}{dr} = -\rho_{(r)} g_{(r)}$$

or
$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$$

constant density: $M_r = \frac{4}{3}\pi r^3 \rho_c$

$$\frac{M_r}{M} = \frac{r^3}{R^3}, \quad r = \left(\frac{M_r}{M}\right)^{\frac{1}{3}} R$$

$$\Rightarrow \frac{dP}{dM_r} = -\frac{GM_r}{4\pi \left(\frac{M_r}{M}\right)^{\frac{4}{3}} R^4} = -\frac{GM}{4\pi R^4}\left(\frac{M_r}{M}\right)^{-\frac{1}{3}}$$

$$\int_{P_c}^{0} dP = -\frac{GM^2}{4\pi R^4}\int_0^M \left(\frac{M_r}{M}\right)^{-\frac{1}{3}} d\frac{M_r}{M}$$

$$P\Big|_{P_c}^{0} = -P_c = -\frac{GM^2}{4\pi R^4}\frac{1}{\left(-\frac{1}{3}+1\right)}\left(\frac{M_r}{M}\right)^{\left(-\frac{1}{3}+1\right)}\Bigg|_0^M$$

$$\Rightarrow \boxed{P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}} = 1.34 \times 10^{15}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4} \text{ dyne cm}^{-2} \quad (1\text{-}42)$$

$$P = P_c\left[1 - \left(\frac{M_r}{M}\right)^{2/3}\right] = P_c\left[1 - \left(\frac{r}{R}\right)^2\right] \quad (1\text{-}41)$$

P_c: lower limit of central pressure, if $\rho_{(r)} \neq \rho_c$

Temperature Distribution of Constant Density Model (1.4.2)

$$P = nkT = \frac{\rho}{\mu} N_A kT$$

$$T = \frac{\mu}{N_A k} \frac{P}{\rho}$$

$$T = T_c \left[1 - \left(\frac{r}{R}\right)^2 \right]$$

$$T_c = \frac{1}{2} \frac{GM}{R} \frac{\mu}{N_A k} \qquad (1.56)$$

$$T_c = 1.15 \times 10^7 \mu \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} K$$

For the Sun, $T_c = 1.15 \times 10^7$ K,

closer to sophisticated numerical model that gives $T_c = 1.5 \times 10^7$ K

Mean Molecular Weights (CH 1-4.1)

μ: average weight per particle in unit of amu

one ^{12}C atom = 12 amu

1 Mole of ^{12}C = 12 gram

$\Rightarrow$ $N_A = 6.022 \times 10^{23}$ / Mole.

μ is the function of time as nuclear fusion changes the abundance

ZAMS (zero-age Main Sequence Star), e.g. Sun

$X \simeq 0.7$. X: mass fraction of hydrogen

$Y \approx 0.3$, Y: mass fraction of helium

$Z \simeq 0.03$ Z: mass fraction of metals

metal: all other elements except H and He.

As fusion goes, $X \downarrow$, $Y \uparrow$.

e.g. $X \simeq 0.5$, $Y \simeq 0.5$.

Half hydrogen and half helium.

* Pure neutral hydrogen

$$\mu = 1$$

* Hydrogen only, fully ionized: 1 proton + 1 electron

$$\mu = \frac{1}{2}$$

For a mixture of particles X_i (or. X, Y, Z)

i: different species, H, He, Metals (C, N, O, Fe)

For each species: ions, electrons, neutrals: $X_{I,i}$, $X_{e,i}$

* Definition of μ: $n = \frac{\rho}{\mu} N_A$

$\Rightarrow$ $\mu = \frac{\rho}{n} N_A$;

$\rho = \rho \sum_i (X_i)$, mass of electrons are negligible.

$n = \sum_i (n_{I,i} + n_{e,i})$

ions, neutrals — electrons for species i

* Mean Molecular Weight per ion.
for a mixture of gas particles.

$$\mu_I = \frac{\rho}{n_I} N_A$$

$$n_I = \sum_i n_{I,i}$$

$$\rho = \sum_i \rho_i = \sum_i \rho X_i \quad \text{— mass fraction}$$

$$n_{I,i} = \frac{\rho X_i}{A_i} N_A.$$

A_i : mass number of species i
or individual molecular weight

^1H , $A_H = 1$

^4He : $A_{He} = 4$

$\langle A_{metals} \rangle = 14$

^{12}C : $A_C = 12$

$$\Rightarrow \mu_I = \frac{\rho N_A}{\sum \frac{\rho X_i N_A}{A_i}} = \frac{1}{\sum_i \frac{X_i}{A_i}} = \left[\sum_i \frac{X_i}{A_i} \right]^{-1} \quad \text{--- (1.40)}$$

The same for a mixture of neutral particles

E.g. For $X = 0.7$ of H, $Y = 0.3$ of He

$$\mu_I = \left[\frac{0.7}{1} + \frac{0.3}{4} \right]^{-1} = 1.29.$$

* Mean Molecular Weight per electron

$$\mu_e = \frac{\rho}{n_e} N_A.$$

$$n_e = \frac{\rho}{\mu_e} N_A$$

$$n_e = \sum_i n_{e,i}$$

$$n_{e,i} = y_i Z_i n_{I,i}$$

Z_i: charge number for species i, $Z=1$ for H, $Z=2$ for He, $Z=6$ for C

y_i: ionization fraction of species i

a function of temperature, i.e., Saha equation.

$y_i = 1$, fully ionization

$$n_{e,i} = y_i Z_i \left(\frac{\rho X_i}{A_i}\right) N_A$$

$$\mu_e = \frac{\rho}{\sum_i y_i Z_i \left(\frac{X_i}{A_i}\right) \rho N_A} \cdot N_A$$

$$\mu_e = \left[\sum_i y_i Z_i \left(\frac{X_i}{A_i}\right)\right]^{-1} \quad \text{------ (1.49)}$$

* Total mean molecular weight

$$n = \frac{\rho}{\mu} N_A$$

$$\mu = \frac{\rho}{n} N_A = \frac{\rho}{n_e + n_I} N_A$$

$$\mu = \frac{1}{\frac{n_e}{\rho N_A} + \frac{n_I}{\rho N_A}} = \left[\frac{1}{\mu_e} + \frac{1}{\mu_I}\right]^{-1} \quad \text{—— (1.50)}$$

or $\frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_I}$

* Exp. Sun-like star $Z \ll 1$, $X + Y = 1$ $Y = 1 - X$, $y_i = 1$

$$\mu_I = \left[\frac{X}{1} + \frac{1-X}{4}\right]^{-1} = \frac{4}{1+3X} \quad \text{------ 1.54}$$

$$\mu_e = \left[1 \cdot 1 \cdot \frac{X}{1} + 1 \cdot 2 \left(\frac{1-X}{4}\right)\right]^{-1} = \frac{2}{1+X} \quad \text{--- 1.53}$$

$$\mu = \frac{4}{3+5X} \quad \text{-- 1.56}$$

$\mu_I = 1.3$; $\mu_e = 1.2$; $\mu = 0.6$