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CHAP. 1. Preliminaries

— Understand the basic property of a star using simple energy principles, instead of solving the full set of differential equations.

— basic properties:

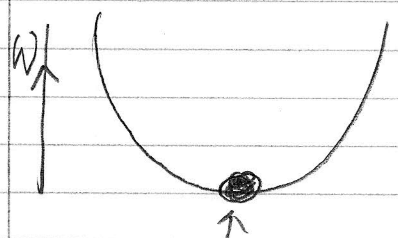
① time scale of protostar : Kelvin-Helmholtz
② time scale of gravitational collapse
③ time scale of main sequence
④ interior temperature

CH 1.2, An energy principle

— derive the hydrostatic equation of motion

$$\boxed{\frac{dP(r)}{dr} = -g(r)\rho(r)}$$

using the minimum energy principle



W: total energy of a star

In hydrostatic equilibrium, $(\delta W)_{ad} = 0$.

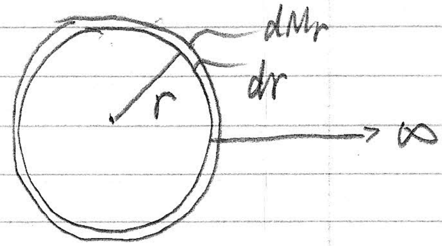
δ: small perturbation operator, e.g. δr displace r by δr

ad: adiabatic change.

$W = U + \Omega$:
- W: total energy of a star
- U: total internal energy of stars
- Ω: total gravitation energy of stars

Ω: total gravitational energy

total energy needed to disperse all elements to infinity.



To move shell dM_r to ∞

$$d\Omega = -\int_r^{\infty} \underbrace{\frac{GM_r}{r'^2} dM_r}_{\text{Gravitational force on shell } dM_r} dr' = -\frac{GM}{r} dM_r$$

To move all shells to ∞ $\quad \Omega = -\int d\Omega$

$$\Omega = -\int_0^M \frac{GM_r}{r} dM_r$$

For contain mass distribution,

$$\Omega = -q \frac{GM^2}{R}, \quad q\text{: an integration factor.}$$

$q = \frac{3}{5}$, if density distribution is ~~not~~ uniform.

U: total internal energy.

internal energy of a collection of gas particles

= kinetic energy of gas particles (translation, rotation, vibration,)

+ potential energy of gas particles ($=0$, for ideal gas)

$$U = \int_M E \, dM_r$$

E: (mass)-specific internal energy : $\underline{\text{erg g}^{-1}}$

\# perturbation analysis

$$W = U + \Omega$$

$$(\delta W)_{ad} = (\delta U)_{ad} + (\delta \Omega)_{ad}$$

$$(\delta W)_{ad} = \left(\delta \int_0^M E \, dM_r\right)_{ad} + \left(-\delta \int_0^M \frac{GM_r}{r} dM_r\right)$$

$$(\delta W)_{ad} = \left(\int_0^M \delta E \, dM_r\right)_{ad} + \left(\int_0^M \frac{GM_r}{r^2} \delta r \, dM_r\right)_{ad}$$

\# Find δE.

Use first law of thermodynamics

$$\underline{dQ} = \underline{dE} + \underline{p\, dV_\rho}$$

heat — internal energy — work done on its surroundings

$V_\rho = \frac{1}{\rho}$. (mass) - specific volume ($cm^3 g^{-1}$)

For adiabatic disturbance : $dQ = 0$, $dE = \delta E$

$$\delta E = -p\, \delta V_\rho$$

$$V_\rho = \frac{1}{\rho} = \frac{4\pi r^2 dr}{dM_r} = \frac{d\left(\frac{4\pi r^3}{3}\right)}{dM_r}$$

$$\delta V_\rho = \delta\left(\frac{d\,\frac{4\pi r^3}{3}}{dM_r}\right) = \frac{d(4\pi r^2 \delta r)}{dM_r} \quad \text{--- (1.14)}$$

$$(\delta U)_{ad} = -\int_0^M p \frac{d(4\pi r^2 \delta r)}{dM_r} dM_r = -\int_0^M p\, d(4\pi r^2 \delta r) \quad \text{--- (1.15)}$$

Integration by parts

$$(\delta U)_{ad} = \cancel{-p \cdot 4\pi r^2 \delta r \Big|_0^M} + \int_0^M 4\pi r^2 \delta r \, dP$$

At $M = 0$, $r = 0$,

$M = M$, $p = 0$

$$(\delta U)_{ad} = \int_0^M \frac{dP}{dM_r} 4\pi r^2 \delta r \cdot dM_r \quad \text{-----}$$

Further $(\delta\Omega)_{ad} = -\delta\int_0^M \frac{GM_r}{r}\, dM_r$

$$(\delta\Omega)_{ad} = \int_0^M \frac{GM_r}{r^2}\,\delta r\, dM_r$$

$$(\delta W)_{ad} = (\delta U)_{ad} + (\delta\Omega)_{ad} = 0$$

$$\Rightarrow \int_0^M \Big(\underbrace{\frac{dP}{dM_r}4\pi r^2 + \frac{GM_r}{r^2}}_{=0}\Big)\,\delta r\, dM_r = 0$$

$$\frac{dP}{dM_r}4\pi r^2 = -\frac{GM_r}{r^2}$$

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \qquad \text{------} \quad (1.16)$$

Or $$\frac{dP}{4\pi r^2 \rho(r)\, dr} = -\frac{GM_r}{4\pi r^4}$$

$$\Rightarrow \frac{dP}{dr} = -\frac{GM_r}{r^2}\rho(r)$$

$$\Rightarrow \frac{dP(r)}{dr} = -g(r)\rho(r) \qquad \text{------} \quad (1.6)$$

The equation of hydrostatic equilibrium

CH 1.3. the Virial Theorem

A classical mechanics approach to describe a large system of particles; not statistical mechanics not fluid dynamics

particle i, m_i, v_i, P_i, F_i

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i \vec{F_i}\cdot\vec{r_i}$$

I: moment of inertial, or angular mass or inertial to rotation

$$I = \sum_i m_i r_i^2$$

K: total ~~(translat~~ kinetic energy

$$K = \sum_i \frac{1}{2} m_i v_i^2$$

$\sum_i \vec{F_i}\cdot\vec{r_i}$: called Virial of Clausius

For a system like star, particles are mutually interacted through gravitational force

$$\vec{F_{ij}} = -\vec{F_{ji}}$$

Gravitational force on particle i, due to particle j

$$\vec{F_{ij}} = -\frac{G M_i M_j}{r_{ij}^3}(\vec{r_i}-\vec{r_j})$$

$$\sum_i \vec{F_i}\cdot\vec{r_i} = \sum_i \Big(\sum_{i<j} \vec{F_{ij}}\cdot\vec{r_i} + \underbrace{\sum_{i>j} \vec{F_{ij}}\,\vec{r_i}}_{\text{flip}(i,j)\ \sum_{i<j} \vec{F_{ji}}\,\vec{r_j}} \Big)$$

$$= \sum_{\substack{i,j \\ i<j}} \vec{F_{ij}}\cdot(\vec{r_i}-\vec{r_j})$$

$$= \sum_{\substack{i,j \\ i<j}} -\frac{G M_i M_j}{r_{ij}^3}(\vec{r_i}-\vec{r_j})\cdot(\vec{r_i}-\vec{r_j}) = -\sum_{\substack{i,j \\ i<j}} \frac{G M_i M_j}{r_{ij}}$$

(Continued)

$-\frac{GM_iM_j}{r_{ij}}$ is the gravitational energy to disperse i, j

$$\text{Virial} = \sum_i \vec{F}_i \cdot \vec{r}_i = \Omega$$

For a gravitational bound system

$$\Rightarrow \boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega}$$ Virial theorem for stars

Prove of Virial Theorem

Consider $\sum_i \vec{p}_i \cdot \vec{r}_i$

$$\frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \frac{d}{dt}\sum_i m_i \vec{v}_i \cdot \vec{r}_i = \frac{d}{dt}\sum_i m_i \frac{d\vec{r}_i}{dt}\cdot \vec{r}_i$$

$$= \frac{1}{2}\frac{d}{dt}\sum_i m_i \frac{dr_i^2}{dt} = \frac{1}{2}\frac{d^2}{dt^2}\underbrace{\sum M_i r_i^2}_{I}$$

$$= \frac{1}{2}\frac{d^2 I}{dt^2}$$

$$\frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \sum_i \left(\frac{d\vec{p}_i}{dt}\right)\cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2\underbrace{\sum_i \frac{1}{2} m_i v_i^2}_{K}$$

$$\Rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i \vec{F}_i \cdot \vec{r}_i$$

Interpret K in a star

relate with other statistical parameters: P, E, U

$$2K = \sum_i m_i v_i^2 = \sum_i \vec{p}_i \cdot \vec{v}_i$$

In a continual medium $\quad 2K = \int_{P,V} \vec{p}\cdot\vec{v}\, n(\vec{p})\, d^3\vec{p}\, dV$

$n(\vec{p})$: number of particles with momentum $\vec{p} \to \vec{p} + d\vec{p}$

On the ~~other~~ hand, the pressure from statistical mechanics

$$P = \frac{1}{3} \int_p \vec{p} \cdot \vec{v} \, n_p \, d^3\vec{p} \quad \text{------} \quad (1.20)$$

Therefore, $$2K = \int_V 3P \, dV \quad \text{------} \quad (1.21)$$

Since $dM_r = \rho \, dV$, or $\rho = \frac{dM_r}{dV} =$

$$2K = 3 \int_M \frac{P}{\rho} \, dM_r$$

$$\Rightarrow \frac{1}{2} \frac{d^2 I}{dt^2} = \int_m \frac{3P}{\rho} \, dM_r + \Omega \quad \text{------} \quad (1.23)$$

CH1-3.1; Global Energetics, W in terms of Ω

From ideal gas ~~of~~ equation of state:

$$P = (\gamma - 1)\rho E$$

γ: adiabatic index, ($P = c\rho^{\gamma}$ in adiabatic process)

: ratio of specific heats ($\gamma = \frac{C_p}{C_v}$)

: $\frac{n+2}{n}$. n: number of freedom

$n=3$, $\gamma = \frac{5}{3}$. monatomic gas

$n=5$, $\gamma = \frac{7}{5}$. diatomic gas.

E: specific internal energy $\mathrm{erg\ g^{-1}}$

ρE: volumetric internal energy $\mathrm{erg\ cm^{-3}}$.

$$\frac{3P}{\rho} = 3(\gamma-1)E$$

$$2K = \int_m \frac{3P}{\rho}\, dM_r = \int_m 3(\gamma-1)E\, dM_r = 3(\gamma-1)U$$

$$\Rightarrow \frac{1}{2}\frac{d^2I}{dt^2} = 3(\gamma-1)U + \Omega \qquad (1.56)$$

Since $W = U + \Omega$ and, $U = -\frac{\Omega}{3(\gamma-1)}$

$d^2I/dt^2 = 0$ for the hydrostatic equilibrium

$$W = U + \Omega = -\frac{\Omega}{3(\gamma-1)} + \Omega$$

$$\boxed{W = \frac{3\gamma-4}{3(\gamma-1)}\Omega} \qquad 1.27$$

For example, $\gamma = \frac{5}{3}$. monotomic ideal gas

$$W = \frac{1}{2}\Omega, \quad U = -\frac{\Omega}{2}, \quad K = U = -\frac{\Omega}{2}$$

$\Rightarrow$ Half of gravitational energy goes to <u>heating</u>, the other <u>radiation</u>

CH 1.3.2 : Kelvin-Helmholtz time scale ; Application Virial

$$W = \frac{3(\gamma-4)}{3(\gamma-1)} \Omega .$$

The time scale of gravitational energy for luminosity

$$\Delta W = \frac{3(\gamma-4)}{3(\gamma-1)} \Delta\Omega .$$

$L = -\frac{dW}{dt}$. change of total energy radiates away.

If $\gamma = \frac{5}{3}$, $\Delta W = \frac{1}{2} \Delta\Omega$

$\Delta U = -\frac{1}{2} \Delta\Omega$

When contracting. $\Delta\Omega < 0$, $\Delta W < 0$, total energy decrease

$\Delta U > 0$, internal energy increases.

$$\Delta\Omega = \Delta\left(q \frac{GM^2}{R}\right) = \frac{q}{2} \frac{GM^2}{R^2} \Delta R$$

$$L = -\frac{dW}{dt} = -\frac{q}{2} \frac{GM^2}{R} \left(\frac{dR/dt}{R}\right) \quad \text{------ (1-30)}$$

$$\frac{dR}{R} = -\frac{1}{\frac{q}{2}\frac{GM^2}{LR}} dt$$

$$R = R_0 e^{-\frac{t}{t_{KH}}}$$

t_{KH}: e-folding time of gravitational contracting of radius size.

$$t_{KH} \approx \frac{q}{2} \frac{GM^2}{LR} .$$

In solar unit

$$t_{KH} \simeq 2\times10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{L}{L_\odot}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-1} \text{years}$$

For Sun, gravitational contracting can't supply the energy

Energy sources of a star

① gravitational energy — protostar

② nuclear energy — main sequence

③ internal energy — white dwarf.

1.33. A Dynamic Time Scale : Virial Theorem Application

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3(\gamma-1)U + \Omega$$

The time scale makes the dynamic adjustments in structure, e.g. star collapse.

Using dimensional analysis

$$\frac{1}{2}\frac{d^2I}{dt^2} = \Omega$$

$$\frac{I}{t_{dyn}^2} \sim \Omega \sim \frac{GM^2}{R}$$

$$\frac{MR^2}{t_{dyn}^2} \sim \frac{GM^2}{R}$$

$$t_{dyn}^2 = \frac{1}{G\frac{M}{R^3}} \sim \frac{1}{G\langle\rho\rangle}$$

$$t_{dyn} = \frac{1}{\left[G\langle\rho\rangle\right]^{\frac{1}{2}}}$$

For sun, $t_{dyn} = 1$ hour.