Lect. 24 ; April 29, 2010 ①

\# plane EM wave properties (CH 23-2)

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\vec{H} = \vec{H}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$\vec{E}_0$, $\vec{H}_0$ are independent of $(\vec{r}, t)$: space-time

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

Eq. 1 : $\nabla\cdot\vec{E} = 0 \Rightarrow i\vec{k}\cdot\vec{E} = 0$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = ik_x E_x + ik_y E_y + ik_z E_z = i(\vec{k}\cdot\vec{E})$$

where $E_x = E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

Therefore $\vec{k}\cdot\vec{E} = 0$ ------ (18)

Eq. 2 Similarly : $\nabla\cdot\vec{H} = 0 \Rightarrow \vec{k}\cdot\vec{H} = 0$ ------ (19)

Eq. 3 $\nabla\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

LHS : $\nabla\times\vec{E} = i\vec{k}\times\vec{E}$

RHS : $-\mu_0 \frac{\partial \vec{H}}{\partial t} = i\omega\mu_0 \vec{H}$; $\frac{\partial \vec{H}}{\partial t} = -i\omega\vec{H}$

$$\Rightarrow \vec{H} = \frac{\vec{k}\times\vec{E}}{\omega\mu_0} \quad \text{------ (21)}$$

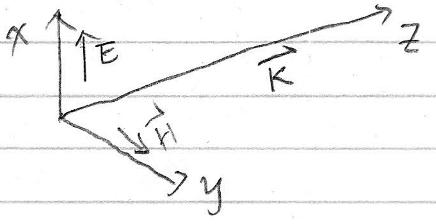
Eq. 4. Similarly $\nabla\times\vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

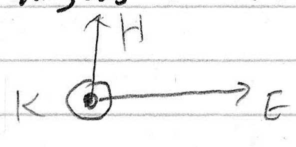
$\cancel{-i\omega\mu_0\vec{H} = i\ldots}$

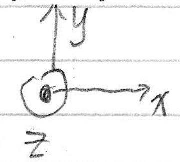
$$i\vec{k}\times\vec{H} = \varepsilon_0(-i\omega)\vec{E}$$

$$\vec{E} = \frac{\vec{H}\times\vec{k}}{\omega\varepsilon} \quad \text{------ (22)}$$

Therefore. $\vec{E}$, $\vec{H}$, $\vec{k}$ are at right angles to one another







plane EM wave properties (CH 23.2). (continued)

Eq. 21 $\Rightarrow$ $H_0 = \dfrac{k}{\omega\mu_0} E_0$

Eq. 22 $\Rightarrow$ $E_0 = \dfrac{k}{\omega\varepsilon} H_0$

Combine the two : $\left(\dfrac{k}{\omega}\right)^2 = \varepsilon\mu_0$

$v = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\varepsilon\mu_0}}$ again, the wave velocity —(27)

Note : Since k, ε, μ_0 are real numbers

$\vec{E}$ and $\vec{H}$ are in phase ; $\vec{E} = 0 \Leftrightarrow \vec{H} = 0$

In free space, $\varepsilon_0 \neq 0$; $\varepsilon_0 \neq \infty$

$\mu_0 \neq 0$, $\mu_0 \neq \infty$

The speed of EM wave is finite , not infinite

Poynting Vector (CH 23.4)

$\vec{S} = \vec{E} \times \vec{H}$: the energy flux, or radiation intensity

From Eq (3) : $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

From Eq (4) : $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$: True in any medium

$$\nabla \cdot \vec{S} = \nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \quad \text{----(54)}$$

Have used the identity: $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{J} \cdot \vec{E} \quad \text{-----(55)}$$

For a linear, isotropic material: $\vec{B} = \mu \vec{H}$, $\vec{D} = \varepsilon \vec{E}$

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} (\vec{H} \cdot \vec{H}) = \frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H})$$

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \varepsilon \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) = \frac{1}{2} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E})$$

$$\Rightarrow \nabla \cdot \vec{S} = \nabla \cdot (\vec{E} \times \vec{H}) = -\frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H}) - \frac{1}{2} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E}) - \vec{J} \cdot \vec{E}$$

or $$\nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = -\vec{J} \cdot \vec{E} \quad \text{-----(56)}$$

where $\vec{S} = \vec{E} \times \vec{H}$: poynting vector

$u = \frac{1}{2} \vec{B} \cdot \vec{H} + \frac{1}{2} \vec{D} \cdot \vec{E}$: energy density

$\vec{J} \cdot \vec{E}$: rate of Joule heating

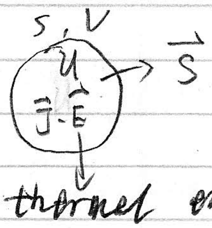
Integral form: $$\underset{①}{\oint_S \vec{S} \cdot d\vec{a}} + \underset{②}{\frac{\partial}{\partial t} \int_V u \, dV} = \underset{③}{-\int \vec{J} \cdot \vec{E} \, dV}$$

S : enclosed surface ; V : volume

① energy per unit time flowing out S

② Total electric and magnetic energy change rate in V

③ Joule heating rate in V, EM energy → thermal energy



Energy Density and Intensity of an EM wave (CH 23.5)

$$u = \frac{1}{2}\vec{D}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{H}$$

$$u = \underbrace{\frac{1}{2}\varepsilon E^2}_{\text{① electric energy}} + \underbrace{\frac{1}{2\mu}B^2}_{\text{② magnetic energy}} \qquad \text{(74)}$$

① $u_E(t) = \frac{1}{2}\varepsilon E_0^2 \cos^2(kz - \omega t)$. electric energy

$$\langle u_E \rangle = \frac{1}{2}\varepsilon E_0^2 \underbrace{\langle \cos^2(kz-\omega t)\rangle}_{=\frac{1}{2}}$$

$$\langle u_E \rangle = \frac{1}{4}\varepsilon E_0^2$$

② Similarly $\langle u_B \rangle = \frac{1}{4\mu} B_0^2$

Because: $\vec{H} = \frac{\vec{k}\times\vec{E}}{\omega\mu_0} \Rightarrow H = \frac{kE}{\omega\mu_0}$

$$\Rightarrow B = \frac{k}{\omega}E = \frac{1}{v}E = \sqrt{\varepsilon\mu}\,E$$

$$\Rightarrow \langle u_E \rangle = \langle u_B \rangle$$

* Energy associated with the electric field is equal to the energy associated with the magnetic energy

→ Total EM energy density

$$\langle u \rangle = \langle u_E \rangle + \langle u_B \rangle = \frac{1}{2}\varepsilon E_0^2 = \frac{1}{2\mu_0}B_0^2 \qquad \text{(77)}$$

* ~~Tot~~ The radiation Intensity

$$I = v\langle u \rangle = \frac{1}{\sqrt{\varepsilon\mu}}\cdot\frac{1}{2}\varepsilon E_0^2$$

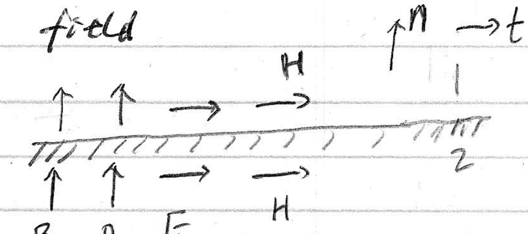
$$I = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0^2 \qquad \text{(78)}$$

In free space, $I = \frac{1}{2}\varepsilon_0 c E_0^2$

c: speed of light in free space

Continuity Conditions (CH 23.8)

consider the electric and magnetic field at the interface of two media



$B_{1n} = B_{2n}$, but $B_{1t} \neq B_{2t}$

$D_{1n} = D_{2n}$, but $D_{1t} \neq D_{2t}$

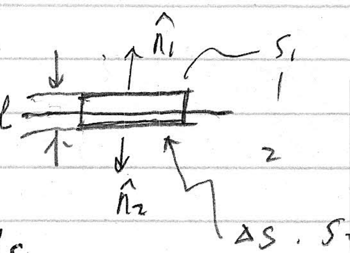
$H_{1t} = H_{2t}$ but $H_{1n} \neq H_{2n}$

$E_{1t} = E_{2t}$, but $E_{1n} \neq E_{2n}$

In the absence of any surface current and surface charge,
the normal component of $\vec{B}$ and $\vec{D}$ are continuous
the tangential component of $\vec{H}$ and $\vec{E}$ are continuous

Eq②:

$\nabla \cdot \vec{B} = 0$



consider a cylinder

$$\int_V (\nabla \cdot \vec{B})\, dV = \oint_S \vec{B} \cdot d\vec{s} = 0$$

$$= \cancel{\oint} \int_{S_1} \vec{B} \cdot ds + \int_{S_2} \vec{B} \cdot ds + \int_{S_3} \vec{B} \cdot ds$$

S_1: upper surface; S_2: lower surface, S_3: curved surface of the cylinder

$l \to 0$, $\to S_3 = 0$

$$\int_{S_1} \vec{B} \cdot ds + \int_{S_2} \vec{B} \cdot d\vec{s} = 0$$

$\Rightarrow \vec{B}_1 \cdot \hat{n}_1 \Delta S + \vec{B}_2 \cdot \hat{n}_2 \Delta S = 0$, $\hat{n}_1 = -\hat{n}_2$

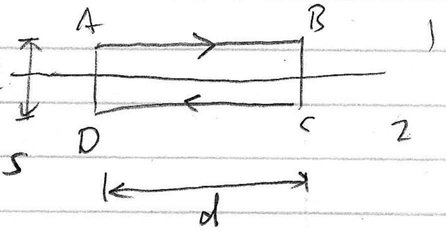
$\Rightarrow B_{1n} = B_{2n}$

Eq③ $\nabla \cdot \vec{D} = 0 \Rightarrow D_{1n} = D_{2n}$; or $D_{1n} - D_{2n} = \underline{\sigma}$ surface charge density

* Continuity Condition (Ch 23.8) (continued)

Consider a rectangle

Eq(3) : $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$



LHS: $\int_S (\nabla \times \vec{E}) \cdot d\vec{s} = \oint_L \vec{E} \cdot d\vec{l}$

L: ABCDA. closed loop; surface S

$= \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \right) \vec{E} \cdot d\vec{l}$

Let l (height of the rectangle) $\to 0$

$\int_S (\nabla \times \vec{E}) \cdot d\vec{s} = \int_{AB} \vec{E} \cdot d\vec{l} + \int_{CD} \vec{E} \cdot d\vec{l} = E_{1t} \cdot d - E_{2t} \cdot d$

RHS: $\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = \frac{\partial \vec{B}}{\partial t} \cdot \int_S ds = \frac{\partial B}{\partial t} \cdot l \cdot d = 0 \quad (l \to 0)$

Therefore $E_{1t} = E_{2t}$

Similarly. Eq(4): $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

$\Rightarrow H_{1t} = H_{2t}$

If presence of surface current

$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \Rightarrow H_{1t} - H_{2t} = \sigma_J$

surface current density