Lect. 23 ; Apr. 23, 2010

Electromagnetic Waves : EM waves (CH23)
Described by Maxwell's equations

A set of four equations (in differential term)

① $\nabla \cdot \vec{D} = \rho$ ---------- (1)

D : electric displacement

ρ : density of free electric charges

② $\nabla \cdot \vec{B} = 0$ ---------- (2)

B : magnetic induction

③ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ---------- (3)

$\vec{E}$: electric field ; $V = E \cdot d$ (voltage)

④ $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ ---------- (4)

$\vec{J}$: electric current $\vec{J} = n_q \vec{U}_q$

$\vec{H}$: magnetic field

where do these equation originate? (CH 23.9)

Eq. ① + ② from "Gauss" Law

Eq. ③ from "Faraday"'s Law

Eq. ④ from "Ampere"'s Law

Eq ①: Gauss's Law (1835):

"The ~~electric~~ electric flux through any closed surface is proportional to the enclosed electric charge"

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \quad ; \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Therefore, $\oint_S \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0}\int_V \rho\, dV$

using the Divergence Theorey: $\oint \vec{E} \cdot d\vec{s} = \int_V (\nabla \cdot \vec{E})\, dv$

$\Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$, since $\vec{D} = \varepsilon_0 \vec{E}$ in free space

$\Rightarrow \nabla \cdot \vec{D} = \rho$

Maxwell's Equations (continued)

Eq (2): Gauss's Law for magnetic flux

$$\nabla \cdot \vec{B} = 0$$

Since there is no magnetic charge, or monopole.

Eq (3): Faraday's Law (1831)

the basic Law for electric generators and motors.

" The induced electromotive force or EMF in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\mathcal{E} = \oint_c \vec{E} \cdot d\vec{l} \quad : \vec{c} \text{ : closed loop}$$

$$\Phi_B = \int_s \vec{B} \cdot d\vec{s} \quad : \vec{s} \text{ : the surface enclosed by } \vec{l}$$

Using Stoke's Theorem

$$\oint \vec{E} \cdot d\vec{l} = \int_s (\nabla \times \vec{E}) \cdot d\vec{s}$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Eq (4): Ampere's Law (1826)

" The amount of magnetic field around a closed loop is proportional to the total amount of electric current enclosed by the loop"

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 \int_s \vec{J} \cdot d\vec{s} \quad ;$$

$$\oint_c \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$\oint_c \vec{B} \cdot d\vec{l} = 2\pi r B \quad \text{If circular symmetric} \quad B = \frac{\mu_0 I}{2\pi r}$$

Using stoke's theorem

$$\oint_c \vec{B} \cdot d\vec{l} = \int_s (\nabla \times \vec{B}) \cdot d\vec{s}$$

Maxwell's Equations (continued)

$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$$

In free space $\vec{B} = \mu_0 \vec{H}$

$$\Rightarrow \nabla \times \vec{H} = \vec{J}$$

There is a problem with this Ampere's Law

$\nabla \cdot (\nabla \times \vec{H}) \equiv 0$, Divergence of curl $\equiv 0$

$\nabla \cdot \vec{J} = 0$: This disobeys the continuity equation that the electric charges have to be conserved

$$\nabla \cdot \vec{J} + \underbrace{\frac{\partial \rho}{\partial t}} = 0$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \underbrace{\frac{\partial}{\partial t} (\nabla \cdot \vec{D})}_{\text{Eq. 1}} = 0$$

$$\Rightarrow \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \text{--- Eq. 4}$$

$\frac{\partial \vec{D}}{\partial t}$: displacement current $\rightarrow$ leads to the revolution.

There are two ways to generate magnetic field.

1. displacement current $\rightarrow$ EM wave
2. electric current

constitutive relations

Four equations and $\vec{E}$, $\vec{D}$, $\vec{H}$, $\vec{B}$, $\vec{J}$

In a linear, isotropic and homogeneous medium

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi \vec{E} \quad \text{--- (5)}$$

$$\vec{B} = \mu \vec{H} = \mu_0 \vec{H} + \vec{M} = \mu_0 \vec{H} + \chi_m \vec{H} \quad \text{--- (6)}$$

$$\vec{J} = \sigma \vec{E} \quad \text{--- (7)}$$

ε: dielectric permitivity ; $\varepsilon = \varepsilon_0 + \chi$; χ: susceptibility

μ: magnetic peameability ; $\mu = \mu_0 + \chi_m$

σ: conductivity.

$(\varepsilon, \mu, \sigma)$ the property of the medium

Further, for $\boxed{\text{dielectric}}$, including free space / vacuum

charge-free, current-free

non-magnetic

$\mu = \mu_0 = 4\pi \times 10^{-7}\ N s^2 c^{-2}$

magnetic peameability in free space

$\sigma = 0$.

$\Rightarrow \vec{J} = 0$

Also $\rho = 0$

The Maxwell's Eqs reduce to

$$\nabla \cdot \vec{D} = 0 \quad \text{--- (11)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{--- (12)}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \text{--- (13)}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \text{--- (14)}$$

The EM wave is determined by "ε"

$\vec{E}$, $\vec{H}$ can be found from the equations

The Wave Equation in a Dielectric (CH 23.3)

Eq.3. $\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$\nabla \times$ Eq.3 $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$; vector identity

$\nabla^2 \vec{E}$: Laplace $\vec{E}$; $\nabla^2 E = (\nabla \cdot \nabla)\vec{E}$

In Cartisian Coordinate : $\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}$

$$\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix}$$

$$\nabla \times \vec{E} = \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

In dielectric $\nabla \cdot \vec{E} = \nabla \cdot \frac{\vec{D}}{\varepsilon_0} = 0$ (Eq.1)

LHS : $\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$

RHS : $-\mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{H}) = -\mu_0 \varepsilon \frac{\partial \vec{E}}{\partial t}$

In which Eq 4. $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$ is used

Therefore $\nabla^2 \vec{E} = \varepsilon \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ ——— (48)

Recall the scalar wave Equation (CH 11.9)

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

In one dimension $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$

General solution : $\psi(x,t) = f(x - vt) + g(x + vt)$

Simple solution : $\psi(x,t) = a \cos[k(x - vt)]$

Therefore, $\vec{E}$ and $\vec{H}$ can be in a wave form : EM wave

$\boxed{v = \frac{1}{\sqrt{\varepsilon \mu_0}}}$; $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ in free space

EM wave (Ch 23.3). (continued)

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\varepsilon_0 = 8.8542 \times 10^{-12}\ C^2 N^{-1} m^{-2}$$

$$\mu_0 = 4\pi \times 10^{-7}\ N s^2 C^{-2}$$

$$c = 2.99794 \times 10^8\ m s^{-1}$$

$$v = \frac{1}{\sqrt{\varepsilon \mu_0}}$$

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}$$

$$v = \frac{\omega}{k}$$

$$k = \frac{\omega}{v} = \omega\sqrt{\varepsilon \mu_0}$$

$\varepsilon \uparrow$, $v \downarrow$, $k \uparrow$, $\lambda \downarrow$. but <u>ω = unchanged</u>