Lect. 18 ; Apr. 6, 2010

Part 4 — Diffraction

App: Diffraction grating ⟹ spectroscopy

Diffraction : ~~interference~~ superposition from an area source,
equivalent to numerous point sources.

Interference : superposition from two point sources, or limited number

Fraunhofer Diffraction (CH 18)

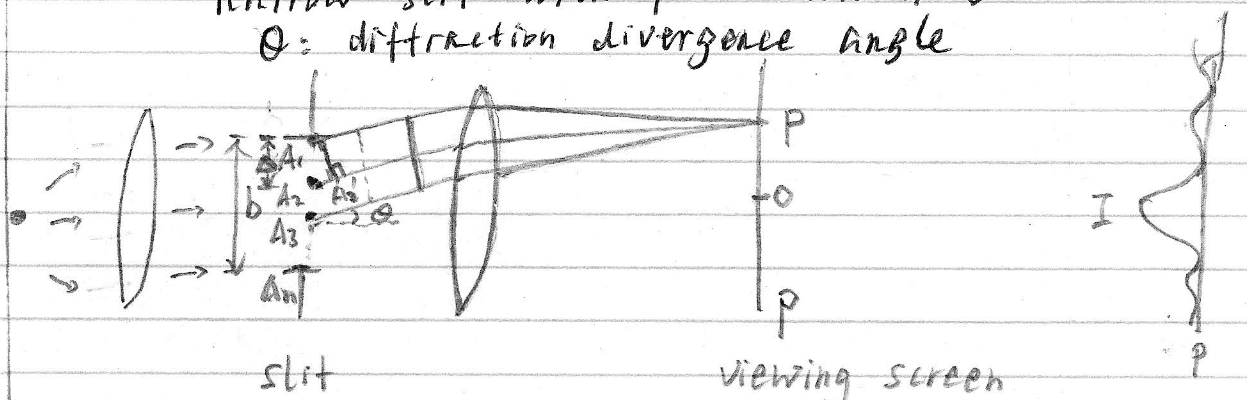
Both sources and viewing screen are at infinity

Fresnel diffraction, either source or screen at finite distance.

Single - slit Diffraction Pattern (CH 18-2)

narrow slit with finite width b

θ : diffraction divergence angle



Each point, e.g. A_1, A_2, A_3, is a source of Huygen's secondary wavelet

Let $\Delta = \overline{A_1A_2} = \overline{A_2A_3}$, and n points in the slit

$b = (n-1)\,\Delta$

Consider the optical path difference between A_1P and A_2P, τ

$$\tau = \tau_{A_1P} - \tau_{A_2P} = A_2A_2' = \Delta \sin\theta$$

The corresponding phase difference

$$\phi = \frac{2\pi}{\lambda}\Delta \sin\theta \qquad \text{---- (2)}$$

At the slit, all points are in phase.

* Single-slit Diffraction (continued)

The resulting electric field at P

From A_1 $\quad E_1 = a\cos\omega t = a e^{i\omega t}$

$A_2 \quad E_2 = a\cos(\omega t + \phi) = a e^{i\omega t} e^{+i\phi}$

$A_3 \quad E_3 = a\cos(\omega t + 2\phi) = a e^{i\omega t} e^{+i2\phi}$

$\vdots$

$A_n \quad E_n = a\cos(\omega t + (n-1)\phi) = a e^{i\omega t} e^{+i(n-1)\phi}$

$$E = E_1 + E_2 + E_3 + \cdots E_n$$

$$E = a e^{i\omega t}(1 + e^{+i\phi} + e^{+i2\phi} + \cdots e^{+i(n-1)\phi})$$

Using the formula of partial sum of series

$$1 + r + r^2 + r^3 \cdots r^{N-1} = \sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad r \neq 1$$

If $N \to \infty$ $\quad \lim_{N\to\infty} \sum_{k=0}^{N-1} r^k = \frac{1}{1-r}$: Taylor expansion

$$(1 + e^{i\phi} + e^{i2\phi} + \cdots e^{i(n-1)\phi}) = \frac{1 - e^{iN\phi}}{1 - e^{i\phi}}$$

Since $e^{i\phi} = e^{i\frac{\phi}{2}} e^{i\frac{\phi}{2}}$

$$\text{Series} = \frac{e^{i\frac{N\phi}{2}}(e^{-i\frac{N\phi}{2}} - e^{i\frac{N\phi}{2}})}{e^{i\frac{\phi}{2}}(e^{-i\frac{\phi}{2}} - e^{i\frac{\phi}{2}})}$$

$$= e^{i(N-1)\frac{\phi}{2}} \cdot \frac{\sin(\frac{N\phi}{2})}{\sin\frac{\phi}{2}}$$

$$\therefore \quad E = a\underbrace{\frac{\sin(\frac{N\phi}{2})}{\sin\frac{\phi}{2}}}_{\text{amplitude}} \cdot \underbrace{e^{i[\omega t + (\frac{N-1}{2})\phi]}}_{\text{sinusoidal part}}$$

; $N\phi$: total phase delay from A_1 to A_N

$\frac{N\phi}{2}$: half phase delay

$$\frac{N\phi}{2} = \frac{N}{2}\cdot\frac{2\pi}{\lambda}\Delta\sin\theta = \frac{\pi}{\lambda} b\sin\theta, \quad \text{since } N\Delta = b$$

$$\frac{\phi}{2} = \frac{\pi}{\lambda}\frac{b}{N}\sin\theta \ll 1$$

Single-slit Diffraction (continued)

$$E = a\frac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)} = a\frac{\sin\left(\frac{\pi b\sin\theta}{\lambda}\right)}{\frac{\pi b\sin\theta}{\lambda}\frac{1}{N}}$$

$\cancel{E = A\sin\beta}$ $$E = A\frac{\sin\beta}{\beta} \quad \text{------} \quad (7)$$

where $\boxed{\beta = \frac{\pi b\sin\theta}{\lambda}}$; $\beta = \frac{N\phi}{2}$: phase delay between edge to center

$A = Na$: the total amplitude at the slit

$I_0 \propto A^2$

The interference pattern at P

$$I = E^2 = I_0\frac{\sin^2\beta}{\beta^2} \quad \text{------} \quad (10)$$

The sinc function

Positions of Maxima and Minima

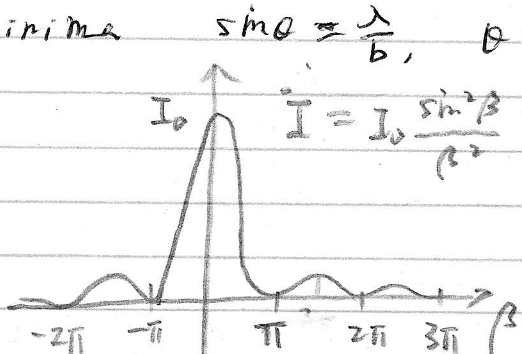
$\theta = 0,\ \beta = 0 \qquad \frac{\sin\beta}{\beta} = 1.\ \ I = I_0$ maximum :

$\theta \neq 0,\ \beta = m\pi \ (m \neq 0),\quad \sin\beta = 0,\ \ I = 0$: minima

minima : $\frac{\pi b\sin\theta}{\lambda} = m\pi$

minima $\quad b\sin\theta = m\lambda \qquad m = \pm1, \pm2, \cdots \qquad (12)$

First minima $\quad \sin\theta = \frac{\lambda}{b},\ \ \theta = \sin^{-1}\left(\frac{\lambda}{b}\right),\ \beta = \pi$



Maxima $\frac{dI}{d\beta} = 0 \Rightarrow \frac{2\sin\beta\cos\beta}{\beta^2} - \frac{2}{\beta}\frac{\sin^2\beta}{\beta^2} = 0$; $\sin\beta\left(\cos\beta - \frac{\sin\beta}{\beta}\right) = 0$

Single - slit Diffraction (continued)

$\tan\beta = \beta$ (maxima)

$\beta = 0$, central maximum $I = I_0$

$\beta = 1.43\pi$, first maximum $I = 4.96\% \, I_0$

$\beta = 2.46\pi$, 2nd maximum $I = 1.68\% \, I_0$

At first maxima $I = I_0 \left(\frac{\sin 1.43\pi}{1.43\pi} \right)^2 = 0.0496 \, I_0$

Exp: $\lambda = 5 \times 10^{-5}$ cm $= 500$ nm

$b = 0.2$ mm

$f = 20$ cm focal length of the convex lens

Location of the first and second minima?

First minimum $\beta = \pi$, $\sin\theta_1 = \frac{\lambda}{b}$

$\theta_1 = \sin^{-1}(\frac{\lambda}{b}) = \sin^{-1}(\frac{5\times10^{-5}}{0.02}) = 2.5 \times 10^{-3}$ rad

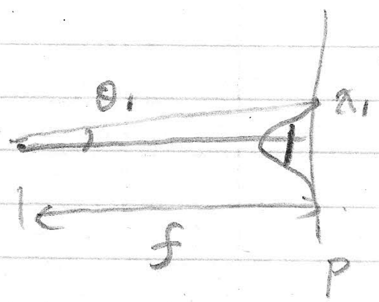
$x_1 = f\theta_1 = 20\text{cm} \times 2.5\times10^{-3} = 0.05$ cm

Second minimum $\beta = m\pi = 2\pi$

$\sin\theta_2 = \frac{2\lambda}{b}$

$\theta_2 = 5.0 \times 10^{-3}$

$x_2 = f\theta_2 = 0.10$ cm.



Width of central maximum

first minima $\theta_1 = \frac{\lambda}{b}$

$x_1 = f\theta_1$

width $\Delta\theta \sim \frac{\lambda}{b}$

$b \downarrow$, $\Delta\theta \uparrow$; $\Delta\theta$: the divergence angle