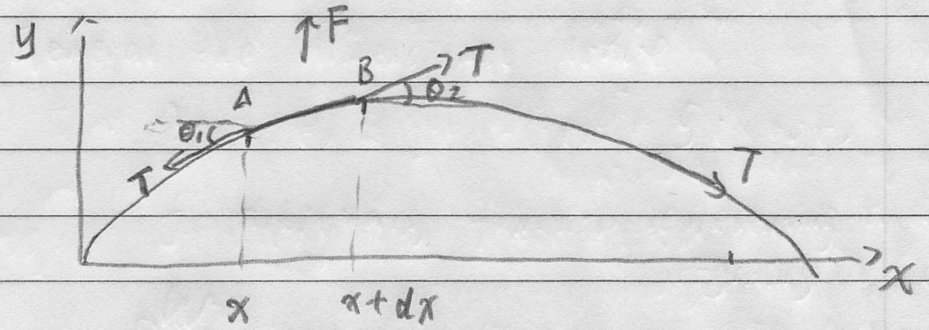
Lect. 8 ; Feb. 18, 2010

* Transverse Vibration of a stretched String (CH 11.6)



A string with tension force T,

Line density ρ : mass per unit length

Consider the segment $\overline{AB}$, the force F in y direction

$$F_A = -T\sin\theta_1 \approx -T\tan\theta_1 = -T\left.\frac{\partial y}{\partial x}\right|_x$$

small vibration

$$F_B = T\sin\theta_2 \approx +T\tan\theta_2 = T\left.\frac{\partial y}{\partial x}\right|_{x+dx}$$

$$F = F_B - F_A = T\underbrace{\left[\left.\frac{\partial y}{\partial x}\right|_{x+dx} - \left.\frac{\partial y}{\partial x}\right|_x\right]}$$

$$= T\underbrace{\frac{\partial^2 y}{\partial x^2}\,dx}$$

Taylor expansion series was used here

$$\left.\frac{\partial y}{\partial x}\right|_{x+dx} \approx \left(\frac{\partial y}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Big|_x dx + \cdots$$

EOM $\quad \Delta m \dfrac{\partial^2 y}{\partial t^2} = T\dfrac{\partial^2 y}{\partial x^2}\,dx$

$$\Delta m = \rho\, dx$$

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{T/\rho}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Therefore $v = \sqrt{\frac{T}{\rho}}$

$T\uparrow,\ v\uparrow$

$\rho\uparrow,\ v\downarrow$

Solution (P115, Eq. 31. in CH 8.2)

$$y_{(x,t)} = \frac{2dL^2}{a(L-a)\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{L}a\right)\cdot\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi v}{L}t\right)$$

* Longitudinal Wave In a Solid (CH 11-7)

$$v = \left(\frac{Y}{\rho}\right)^{\frac{1}{2}}$$

Y ; Young's modulus

$$Y = \frac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = \frac{(F/A)}{\left(\frac{\text{Increase in length}}{\text{Original length}} = \frac{\partial \xi}{\partial x}\right)}$$

ρ = mass density

* Longitudinal wave In a Gas (CH 11-8)

$$v = \left(\frac{\gamma P}{\rho}\right)^{\frac{1}{2}}$$

$\gamma = \frac{c_p}{c_v}$; ratio of the specific heats or adiabatic index

$\gamma = \frac{5}{3}$ for monatomic ideal gas

P : gas pressure ; ρ : gas mass density

★ Electromagnetic Wave (CH 23)

$$v = \frac{1}{\sqrt{\varepsilon \mu}}$$

ε : dielectric permittivity

μ : magnetic permeability

Group Velocity & Pulse Dispersion (CH 6)

For a sinusoidal wave with a single frequency ω

$$\psi(z,t) = A\cos[\omega t - kz]$$

$$\psi(z,t) = A\cos[k(z - vt)] \; ; \quad \omega = kv$$

v is the phase velocity at ω ; $v = \frac{\omega}{k}$

For a non-monochromatic wave with multiple (or continuous) frequency $A(\omega)$ the wave moves at a group velocity $v_g(\omega)$ at ω

$$v_g \,? \qquad v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega}$$

two monochromatic plane waves : $\omega + \Delta\omega$; $\omega - \Delta\omega$

$\Delta\omega \ll \omega$

$$\psi_1(z,t) = A\cos[(\omega + \Delta\omega)t - (k + \Delta k)z]$$

$$\psi_2(z,t) = A\cos[(\omega - \Delta\omega)t - (k - \Delta k)z]$$

Superposition $\psi_1 + \psi_2$

$$\psi_1(z,t) = A\cos[(\omega t - kz) + (\Delta\omega t - \Delta k z)]$$

$$= A\cos(\omega t - kz)\cos[(\Delta\omega)t - (\Delta k)z]$$

$$\cancel{- A\sin(\omega t - kz)\sin[(\Delta\omega)t - (\Delta k)z]}$$

$$\psi_2(z,t) = A\cos[(\omega t - kz) - (\Delta\omega t - \Delta k z)]$$

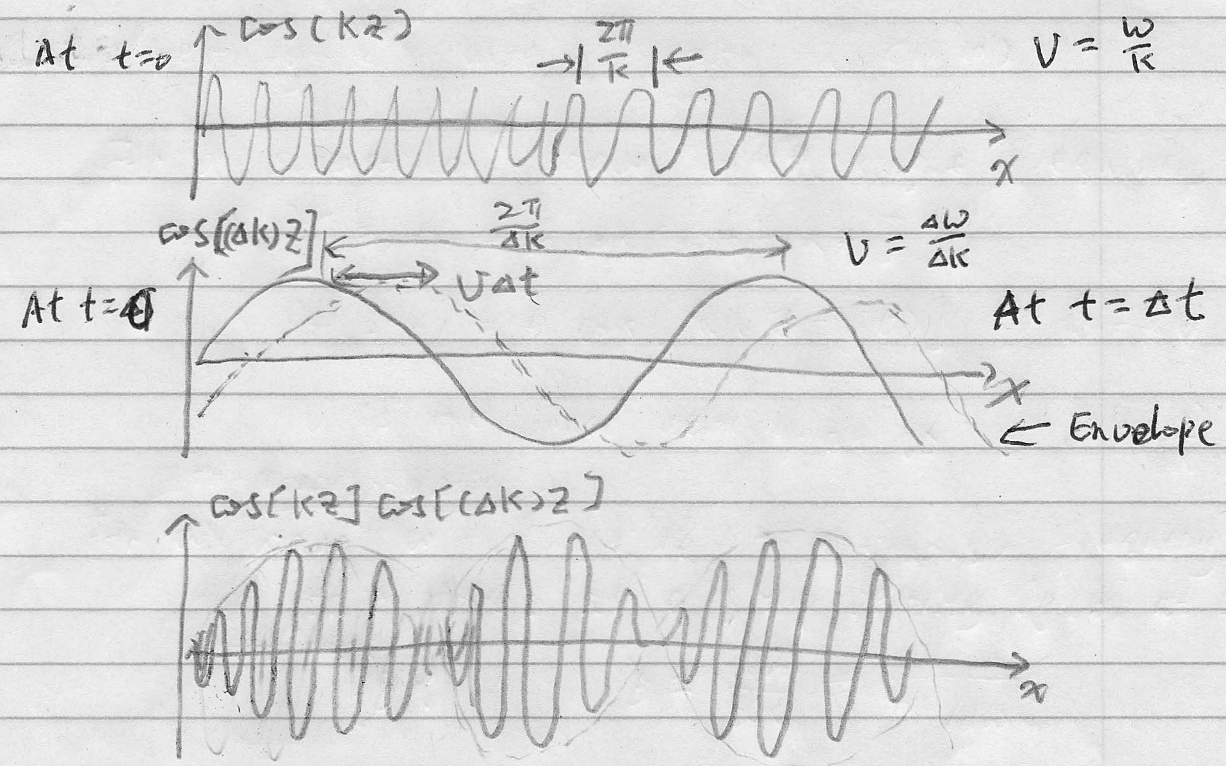
$$= A\cos(\omega t - kz)\cos[(\Delta\omega)t - (\Delta k)z]$$

$$\cancel{+ A\sin(\omega t - kz)\sin[(\Delta\omega)t - (\Delta k)z]}$$

$$\psi = \psi_1 + \psi_2 = 2A\cos[\omega t - kz]\cos[(\Delta\omega)t - (\Delta k)z]$$

$$v_p = \frac{\omega}{k} \qquad\qquad v_g = \frac{\Delta\omega}{\Delta k}$$

rapidly varying wave $\qquad$ slowly varying envelope



The slowly varying envelope moves at the speed of $\frac{\Delta\omega}{\Delta k}$

$$v_g = \frac{\Delta\omega}{\Delta k} \quad \text{the group velocity}$$

$\Delta\omega \to dk$ $\quad v_g = \frac{d\omega}{dk}$

For a frequency-mixed wave. "ω" is an independent variable

In a dielectric medium, refractive index n: $n(\omega)$

e.g. $n(\omega) = 1 + \frac{Nq^2}{m\varepsilon_0\omega_0^2}\left[1 - \frac{\omega^2}{\omega_0^2}\right]^{-1}$ (Eq. 68 in CH7)

$$k(\omega) = \frac{\omega}{v(\omega)} = \frac{\omega}{c} \cdot \frac{c}{v(\omega)} = \frac{\omega}{c} n(\omega)$$

Thus $\cancel{v_g}$ $\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n(\omega) + \omega\frac{dn(\omega)}{d\omega}\right]$

In free space, $n(\omega) = 1$ constant

$v_g = c$

$v_p = c$

It is customary to express in terms of free-space wavelength λ_0

In different medium, ω is the same, λ is different

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\lambda_0/c} = \frac{2\pi c}{\lambda_0} \; ; \quad d\omega = -\frac{2\pi c}{\lambda_0^2}\, d\lambda_0$$

$$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega} = -\frac{\lambda_0^2}{2\pi c}\frac{dn}{d\lambda_0}$$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{1}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}\right]$$

Group index n_g: $\quad n_g = \frac{c}{v_g} = n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}$

For a dielectric medium: $\omega \uparrow$, $\lambda_0 \downarrow$, $n \uparrow$, $\frac{dn}{d\lambda_0} < 0$

Thus $\quad n_g > n(\lambda_0)$: ~~Group~~

$v_g < v_p$: group velocity smaller than phase velocity in a medium