Lect 04 ; Jan. 28, 2010

Rayleigh Scattering (CH 7.6)

Another application of forced vibration

scattering efficient γ : $\gamma \propto \frac{1}{\lambda^4} \propto \omega^4$ --- (63)

scattering by atoms. ~~in~~ dielectric material

Explain why sky is blue. $\frac{\gamma_{blue}}{\gamma_{red}} \sim 10$

and sunset is red.

Scattering particles are smaller than wavelength

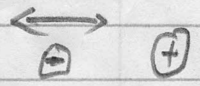
* Tyndall scattering : scattering particles are Larger than wavelength ;
  scattering is uniform in all wavelength
  Explain why cloud is white

* ~~Thomp~~ Thomson scattering : by free electrons

* Compton scattering : X-ray with electrons

* Rayleigh scattering : ~~of~~
  oscillating electric dipole radiates energy

$$P = \frac{q^2}{m(\omega_0^2 - \omega^2)} E$$



$$p = \frac{q^2 E_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = p_0 \cos \omega t$$

Radiation energy by p (CH 23.4.1), the power $\bar{P}$

$$\bar{P} = \frac{p_0^2}{12 \pi \varepsilon_0 c^3} \omega^4 \quad \text{———} \quad (97)$$

$\bar{P} \propto \omega^4$.

Fourier Series (CH8)
Fourier Integral

Fourier's theorem: any periodic vibration can be expressed as a sum of the sine and cosine functions whose frequency increases in the ratio of natural numbers

Periodic function: $f(t+nT) \equiv f(t)$, $n=0, \pm1, \pm2 \cdots$ --- (1)
T: the period

Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi}{T}t\right) \text{ --- (2)}$$

Because $\omega = \frac{2\pi}{T}$, ω: angular frequency

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin n\omega t \text{ --- (2)}$$

ω: also called fundamental frequency
$n\omega$: called harmonics , n: harmonic number

* coefficients a_n, b_n can be calculated from $f(t)$
Different Fourier components are orthogonal

$$\int_{t_0}^{t_0+T} \underline{\cos n\omega t}\ \underline{\cos m\omega t}\, dt = \begin{cases} 0 & \text{if } m \neq n \\ \frac{T}{2} & \text{if } m = n \end{cases} \quad (4)$$

From trigonometry

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$\alpha = n\omega t$, $\beta = m\omega t$

$$\int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t\, dt = \frac{1}{2}\int_{t_0}^{t_0+T}\left[\cos(n+m)\omega t + \cos(n-m)\omega t\right] dt$$

$$= \frac{1}{2}\left[\frac{1}{(n+m)\omega} \sin(n+m)\omega t \Big|_{t_0}^{t_0+T} + \frac{1}{(n-m)\omega} \sin(n-m)\omega t \Big|_{t_0}^{t_0+T}\right.$$

$$\sin[\omega(t+T)] = \sin[\omega t + \omega T] = \sin[\omega t + 2\pi] = \sin\omega t$$

Therefore, if $n \neq m$

$$\int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t = 0$$

If $n = m$

$$\int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t = \frac{1}{2}\int_{t_0}^{t_0+T} [\underbrace{\cos 2n\omega t}_{=0} + 1]dt$$

$$= \frac{1}{2}\int_{t_0}^{t_0+T} dt = \frac{T}{2}$$

~~$a_0 = \int_{t_0}^{t_0+T} f(t)$~~

To obtain a_0, integrate (2) with dt from $t_0 \to t_0+T$

$$\int_{t_0}^{t_0+T} f(t)\,dt = \frac{1}{2}a_0 \int_{t_0}^{t_0+T} dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega t\,dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega t\,dt$$

$$= \frac{1}{2}a_0 T + 0 + 0$$

$$\Rightarrow a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

To obtain a_n, integrate with $\cos n\omega t$

$$\int_{t_0}^{t_0+T} f(t) \cos n\omega t\,dt = \cancel{\frac{1}{2}a_0 \int \cos n\omega t\,dt} + \sum_{m=1}^{\infty} a_m \underbrace{\int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t\,dt}_{= a_n \cdot \frac{T}{2}}$$

$$+ \cancel{\sum_{m=1}^{\infty} b_n \int_{t_0}^{t_0+T} \cos n\omega t \sin m\omega t\,dt}$$

$$\Rightarrow a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \cos n\omega t\,dt \qquad n = 0, 1, 2, \ldots$$

Note $\int_{t_0}^{t_0+T} \cos n\omega t \sin m\omega t\,dt = 0$, for all n and m

$$\int_{t_0}^{t_0+T} \sin n\omega t \sin m\omega t\,dt = \begin{cases} 0, & \text{if } n \neq m \\ T/2, & \text{if } m = n \end{cases}$$

Similarly

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t) \sin n\omega t \, dt \qquad n=0, 1, 2, 3$$

$$b_0 = 0 \qquad \text{when } n=0$$

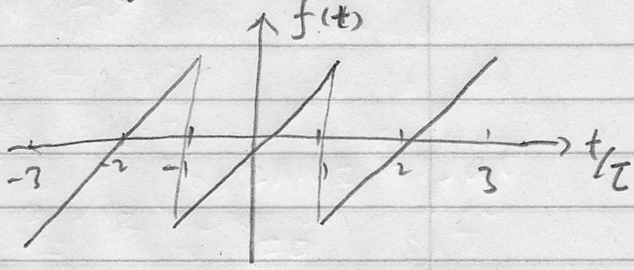
* Example $f(t)$ is a sawtooth function

within a period

$$f(t) = t \qquad -\tau < t < \tau$$

$$f(t+2n\tau) = f(t)$$

period $T = 2\tau$



$f(t)$ is an odd function

$$a_n = \frac{2}{T}\int_{-\tau}^{\tau} \underbrace{f(t)}_{odd} \underbrace{\cos n\omega t}_{even} \, dt \Rightarrow \text{odd } (-\tau, \tau) \qquad = 0$$

$$= \frac{2}{T}\int_{-\tau}^{\tau} t \cos n\omega t \, dt$$

$$= \frac{2}{T}\frac{1}{n\omega}\int_{-\tau}^{\tau} t \, d \sin n\omega t$$

$$= \frac{2}{T}\frac{1}{n\omega}\left[-\int_{-\tau}^{\tau} \sin n\omega t \, dt + \cancel{t \sin n\omega t \Big|_{-\tau}^{\tau}} = 0\right.$$

$$= \frac{2}{T}\frac{(-1)}{n\omega}\left(-\frac{1}{n\omega}\right)\int_{-\tau}^{\tau} d \cos n\omega t$$

$$= + \frac{2}{T}\frac{1}{n^2\omega^2} \cos n\omega t \Big|_{-\tau}^{\tau}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\tau} = \frac{\pi}{\tau}$$

$$\cos n\omega t \Big|_{-\tau}^{\tau} = \cos\left(n \cdot \frac{\pi}{\tau} \tau\right) - \cos\left(-n\frac{\pi}{\tau} \cdot \tau\right)$$

$$= \cos(n\pi) - \cos n\pi = 0$$

$$b_n = \frac{2}{T}\int_{-\tau}^{\tau} \underbrace{f(t)}_{odd} \underbrace{\sin n\omega t}_{odd} \, dt \text{ — even} \qquad = \frac{4}{T}\int_0^{\tau} t \sin n\omega t \, dt$$

$$b_n = \frac{4}{T}\int_0^{t} \frac{-1}{n\omega} t \, d \cos n\omega t$$

$$b_n = \frac{4}{T}\left(\frac{-1}{n\omega}\right)\left[\underbrace{t\cos n\omega t\,\Big|_0^{\tau}}_{\substack{\tau\cos(n\cdot\frac{\pi}{\tau}\cdot\tau)-0 \\ \parallel \\ \tau\cos n\pi}} - \underbrace{\int_0^{\tau}\cos n\omega t\,dt}_{\substack{-\frac{1}{n\omega}\sin n\omega t\,\Big|_0^{\tau} \\ \sin(n\cdot\frac{\pi}{\tau}\cdot\tau)-0 \\ \parallel \\ \cancel{\sin(n\pi)}-0=0 \\ 0}}\right]$$

$$b_n = \frac{4}{2\cancel{\tau}}\,\frac{(-1)}{n\cancel{\omega}\frac{\pi}{\tau}}\,\tau\cos n\pi$$

$$b_n = -\frac{2\tau}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2\tau}{n\pi}$$

because $\cos n\pi = \begin{cases} 1 & n=0,2,4\ldots \text{ even} \\ -1 & n=1,3,5. \text{ odd} \end{cases}$

Therefore

$$f(t) = \frac{2\tau}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\omega t$$

$$f(t) = \frac{2\tau}{\pi}\left(\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t \cdots\right)$$

* plot: use "online Function Grapher"

http://www.walterzorn.com/grapher/graph_e.htm

$\tau = 1$, $\omega = \frac{\pi}{\tau} = \pi$.

$X_{min} = -1$, $X_{max} = 1$, $y_{min} = -1.5$, $y_{max} = 1.5$

n=1 sin(pi*x);

n=2 $-\frac{1}{2}$sin(2*pi*x)

n=3 $+\frac{1}{3}$sin(3*pi*x)

n=4 $-\frac{1}{4}$sin(4*pi*x)

n=5 $+\frac{1}{5}$sin(5*pi*x)