PHYS 306 Spring 2010
Wave Motion and Electromagnetic Radiation

Test 1 - Solution

March 2, 2010

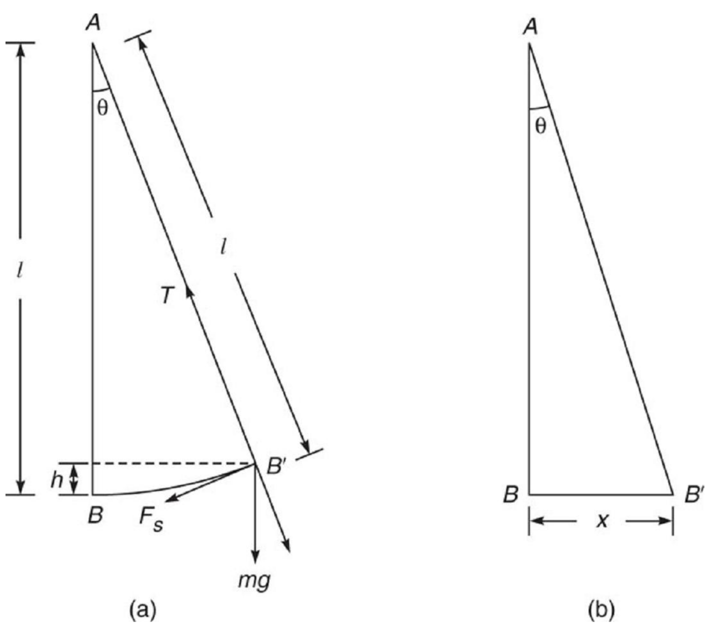
Answer each of the following four questions.

1. (25pts) Consider the simple harmonic motion of a pendulum. The bob of the pendulum of mass M is attached to a string of length L.
(1) Show the equation of motion of the pendulum.
(2) Show that the period of the pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(3) If the string is one meter long, what is the period? and what is the angular frequency?

Answer:



(1) Consider the configuration above. The motion of the bob is caused by the gravitational force projected along the direction of the motion. The equation of motion (EOM) is

$$F = ma$$

$$-mg\sin\theta = m\frac{d^2x}{dt^2}$$

Since θ is small, the motion is approximated as a straight line along the x axis, and $\sin\theta$ = x/L, therefore

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x$$

(2) The general solution of the equation $\frac{d^2x}{dt^2} = -\omega^2 x$ is

$x = a\cos(\omega t + \phi_0)$. Apparently

$\omega = \sqrt{\frac{\mathrm{g}}{\mathrm{L}}}$, the angular frequency

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\mathrm{L}}{\mathrm{g}}}$$

(3)

$$\omega = \sqrt{\frac{\mathrm{g}}{\mathrm{L}}} = \sqrt{\frac{9.8}{1}} = 3.1\ \mathrm{rad/sec}$$

$$T = 2\pi\sqrt{\frac{\mathrm{L}}{\mathrm{g}}} = 2\times 3.14\times\sqrt{\frac{1}{9.8}} = 2.0\ \mathrm{sec}$$

2. (25pts) Consider the periodic sawtooth function of the form

$f(t) = t \qquad$ for $-\tau < t < \tau$

$f(t + 2n\tau) = f(t)$

with the period $T = 2\tau$,

and the *fundamental-frequency* $\omega = \pi/\tau$

Show that the above function can be expanded in a Fourier series of the form

$$f(t) = \frac{2\tau}{\pi}(\sin \omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - .....)$$

Answer:

The discrete Fourier series is defined as

$$\text{f(t)} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t), \text{ where}$$

$$\text{a0} = \frac{2}{\text{T}}\int_{\text{t0}}^{t0+T} f(t)dt$$

$$\text{a}_\text{n} = \frac{2}{\text{T}}\int_{\text{t0}}^{t0+T} f(t)\cos(n\omega t)dt$$

$$\text{b}_\text{n} = \frac{2}{\text{T}}\int_{\text{t0}}^{t0+T} f(t)\sin(n\omega t)dt$$

Consider the integration from $-\tau$ to τ. Since f(t) is an odd function,

$\text{a}_0 = 0$, and $\text{a}_\text{n} = 0$

On the other hand, the integration is over an even function for bn

$$\text{b}_\text{n} = \frac{2}{2\tau}\int_{-\tau}^{+\tau} t\sin(n\omega t)dt = \frac{2}{\tau}\int_0^{+\tau} t\sin(n\omega t)dt$$

$$\text{b}_\text{n} = \frac{2}{\tau}\int_0^{+\tau} t(-\frac{1}{n\omega})d\cos(n\omega t) = \frac{-2}{n\omega\tau}[t\cos(n\omega t)\Big|_0^{\tau} - \int_0^{+\tau}\cos(n\omega t)dt]$$

$$\text{b}_\text{n} = \frac{-2}{n\omega\tau}[t\cos(n\frac{\pi}{\tau}\tau) - \frac{1}{n\omega}\sin(n\omega t)\Big|_0^{\tau}]$$

$$\text{b}_\text{n} = \frac{-2\tau}{n\pi}\cos(n\pi) = \frac{2\tau}{\pi}\frac{(-1)^{n+1}}{n}$$

Therefore,

$$f(t) = \frac{2\tau}{\pi}(\sin \omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - .....)$$

3. (25pts) The displacement function associated with a monochromatic wave is given by

$$y(x,t) = 3.0\cos(4.0x - 2.0t)$$

where x and y are measured in meters and t in seconds.
(1) What is the spatial frequency k? and what is the angular frequency ω?
(2) Calculate the wavelength λ, and period T?
(3) Calculate the propagation velocity of the wave? Which direction does the wave move?

Answer:
(1)
The general format of a wave function (initial phase is zero) is

$$y(x,t) = a\cos(kx - \omega t)$$

Therefore,

$$\mathrm{k} = 4.0\ \mathrm{m}^{-1}$$

$$\omega = 2.0\ \mathrm{rad/sec}$$

(2)

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\times 3.14}{4} = 1.57\ \mathrm{m}$$

$$\omega = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega} = \frac{2\times 3.14}{2.0} = 3.14\ \mathrm{s}$$

(3)

$$v = \frac{\omega}{k} = \frac{\lambda}{T}$$

$$v = \frac{2.0}{4.0} = 0.5\ \mathrm{m/s}$$

The velocity is along the + x direction

4. (25pts) An ionized gas or plasma is a dispersive medium for EM waves. Given the dispersion relation

$$\omega^2 = \omega_p^{\ 2} + c^2 k^2$$

where ω_p is a constant, the "plasma frequency", and c is the speed of light in vacuum.
Determine an expression for the group velocity in this medium in term of the variable ω?

Answer:

According to the defintion of the group velocity

$$v_g = \frac{dw}{dk}$$

Differentiate the equation $\omega^2 = \omega_p^{\ 2} + c^2 k^2$

$$2\omega d\omega = 2c^2 k dk$$

$$v_g = c^2 \frac{k}{\omega}$$

Further, $k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^{\ 2}}$

$$v_g = c\sqrt{1 - (\frac{\omega_p^{\ 2}}{\omega^2})}$$