PHYS 306 Spring 2010
Wave Motion and Electromagnetic Radiation

Homework Assignment - Solution

HW#5
Assignment Date: Feb. 23, 2010
Due Date: Mar. 2, 2010

1. (30pts) Starting from the definition of the group velocity, show that

$$\frac{1}{v_g} = \frac{1}{c}[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}]$$

Answer:
The first step is to express v_g in terms of ω according to the definition, the second step is to replace ω with free space wavelength λ0. Note that both ω and λ0 are intrinsic to the light emitted, independent of medium. On the other hand, group velocity v_g and wave number k are dependent on medium

$$v_g = \frac{d\omega}{dk}; \ \frac{1}{v_g} = \frac{dk}{d\omega}$$

$$k = \frac{\omega}{v(\omega)} = \frac{\omega}{c} n(\omega)$$

$$\frac{1}{v_g} = \frac{1}{c}[n(\omega) + \omega \frac{dn(\omega)}{d\omega}]$$

$$\text{Further, } \omega = \frac{2\pi}{\mathrm{T}} = \frac{2\pi}{(\lambda_0 / c)} = \frac{2\pi c}{\lambda_0}$$

$$d\omega = -\frac{2\pi c}{\lambda_0{}^2} d\lambda_0$$

Plugging - in ω and dω,

$$\frac{1}{v_g} = \frac{1}{c}[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}]$$

2. (30pts) Let

$$n(\lambda_0) = n_0 + A\lambda_0$$

(1) Derive the expression for phase velocity
(2) Derive the expression for group velocity
(3) Derive the expression for the dispersion coefficient

Answer:

(1)

$$v_P = \frac{c}{n} = \frac{c}{n_0 + A\lambda_0}$$

(2)

$$\frac{1}{v_g} = \frac{1}{c}[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}]$$

$$\frac{1}{v_g} = \frac{1}{c}[n_0 + A\lambda_0 - \lambda_0(A)] = \frac{n_0}{c}$$

$$v_g = \frac{c}{n_0}$$

(3) Dispersion coefficient

$$\mathrm{D_m} = \frac{\Delta\tau_\mathrm{m}}{\mathrm{L}\Delta\lambda_0} = -\frac{1}{\lambda_0 c}(\lambda_0{}^2 \frac{d^2 n}{d\lambda_0{}^2})$$

$$\mathrm{D_m} = 0$$

Therefore, in this medium, the phase velocity depends on the free space wavelength, the group velocity is a constant, and the dispersion coefficient is zero.

3. (40pts) For pure silica we may assume the empirical formula

$$n(\lambda_0) = 1.451 - 0.003(\lambda_0^{\ 2} - \frac{1}{\lambda_0^{\ 2}})$$

where λ_0 is measured in μm

(1) Calculate the phase velocity at 0.8 μm
(2) Calculate the group velocity at 0.8 μm
(3) Calculate the dispersion coefficient at 0.8 μm
(4) Considering a LED source emitting a pulse of light of wavelength at 0.8 μm and have a spectral width of 50 nm, what is the broadening time of the pulse over a distance of 2 km.
(5) Calculate the wavelength with zero dispersion.

Note: Please pay attention to the units.

Answer:

(1) At $\lambda_0 = 0.8\ \mu\text{m}$

$n = 1.454$

$$v_P = \frac{c}{n} = \frac{2.998 \times 10^8 \text{ m/s}}{1.454} = 2.062 \times 10^8 \text{ m/s}$$

(2)

$$\frac{1}{v_g} = \frac{1}{c}[n(\lambda_0) - \lambda_0 \frac{dn}{d\lambda_0}]$$

$$\frac{dn}{d\lambda_0} = -0.003(2\lambda_0 + \frac{2}{\lambda_0^{\ 3}}) = -1.364 \times 10^{-2}\ \ \mu\text{m}^{-1}$$

$$\frac{1}{v_g} = \frac{1}{c}[1.454 - 0.8 \times (-1.364 \times 10^{-2})] = \frac{1.465}{c}$$

$$v_g = \frac{2.998 \times 10^8 \text{ m/s}}{1.465} = 2.046 \times 10^8 \text{ m/s}$$

(3) Dispersion coefficient

$$\mathrm{D_m} = -\frac{1}{\lambda_0 c}(\lambda_0^{\ 2}\frac{d^2 n}{d\lambda_0^{\ 2}})$$

$$\frac{dn}{d\lambda_0} = -0.003(2\lambda_0 + \frac{2}{\lambda_0^{\ 3}})$$

$$\frac{d^2 n}{d\lambda_0^{\ 2}} = -0.003(2 - \frac{6}{\lambda_0^{\ 4}}) = 3.795 \times 10^{-2} \quad \mu\mathrm{m}^{-2}$$

$$\mathrm{D_m} = -\frac{1}{\lambda_0 c}(\lambda_0^{\ 2}\frac{d^2 n}{d\lambda_0^{\ 2}})$$

$$\mathrm{D_m} = -\frac{1}{(0.8\mu m)(2.998 \times 10^8 \ \mathrm{m/s})}[(0.8\mu m)^2 \times 3.795 \times 10^{-2} \quad \mu\mathrm{m}^{-2}]$$

$$\mathrm{D_m} = -1.013 \times 10^{-10} \ \mathrm{s} \ \ \mu m^{-1} \ \mathrm{m}^{-1}$$

(4) The dispersion broadening time relates with the dispersion coefficent as

$$\Delta\tau_{\mathrm{m}} = \mathrm{L}\Delta\lambda_0 \mathrm{D_m}$$

$$\Delta\tau_{\mathrm{m}} = (2km)(50nm)(-1.013 \times 10^{10} \ \mathrm{s} \ \ \mu m^{-1} \ \mathrm{m}^{-1})$$

$$\Delta\tau_{\mathrm{m}} = (2 \times 10^3 m)(50 \times 10^{-3} \mu m)(-1.013 \times 10^{-10} \ \mathrm{s} \ \ \mu m^{-1} \ \mathrm{m}^{-1})$$

$$\Delta\tau_{\mathrm{m}} = -1.013 \times 10^{-10} \ \mathrm{s} = \text{-10.13 ns}$$

Therefore, the pulse shrinks by 10.13 ns in this particular material

(5) Zero dispersion means

$$\mathrm{D_m} = -\frac{1}{\lambda_0 c}(\lambda_0^{\ 2}\frac{d^2 n}{d\lambda_0^{\ 2}}) = 0, \ \text{or}$$

$$\frac{d^2 n}{d\lambda_0^{\ 2}} = -0.003(2 - \frac{6}{\lambda_0^{\ 4}}) = 0$$

$$\lambda_0 = 1.316 \ \mu\mathrm{m}$$

The fiber optic communication is optimal at this wavelength