PHYS 306 Spring 2010
Wave Motion and Electromagnetic Radiation

Homework Assignment

HW#3
Assignment Date: Feb. 9, 2010
Due Date: Feb. 16, 2010

1. (Adopted from Example 9.2 in Ghatak book). For a rectangle function given as

$$f(x) = rect(\frac{x}{a}\} = \begin{cases} +1 & \text{for } |x| < \frac{1}{2}a \\ 0 & \text{for } |x| > \frac{1}{2}a \end{cases}$$

(1) show that its Fourier transform is

$$F(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k}$$

(2) make a schematic plot of f(x) and F(k) respectively (hand drawing is acceptable). In the drawing, indicate the approximate (or characteristic) width of the function.

Answer
(1) Using the Fourier Transform
with the factor $1/\sqrt{2\pi}$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \exp(-ikx) dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{+a/2} \exp(-ikx) dx = \frac{1}{\sqrt{2\pi}} \frac{1}{(-ik)} [\exp(-ikx)]_{-a/2}^{a/2}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{(-ik)} [\exp(-ika/2) - \exp(ika/2)]$$

Use the complex exponential identify

$$\exp[ix] = \cos(x) + i\sin(x)$$

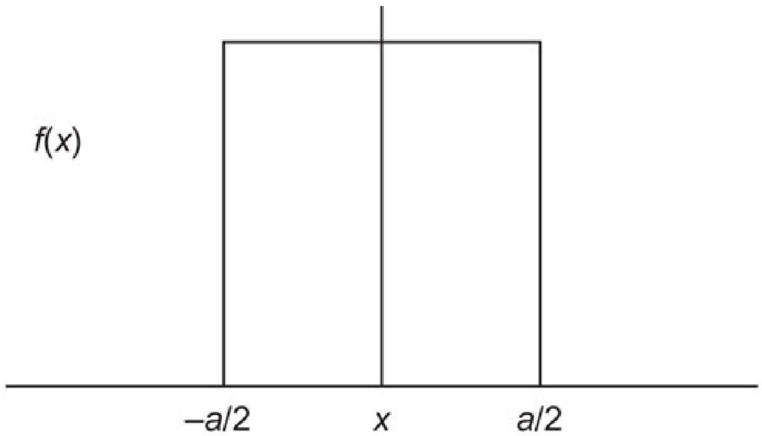
$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{(-ik)} [-2i\sin(ka/2)]$$

$$F(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(ka/2)}{k}$$

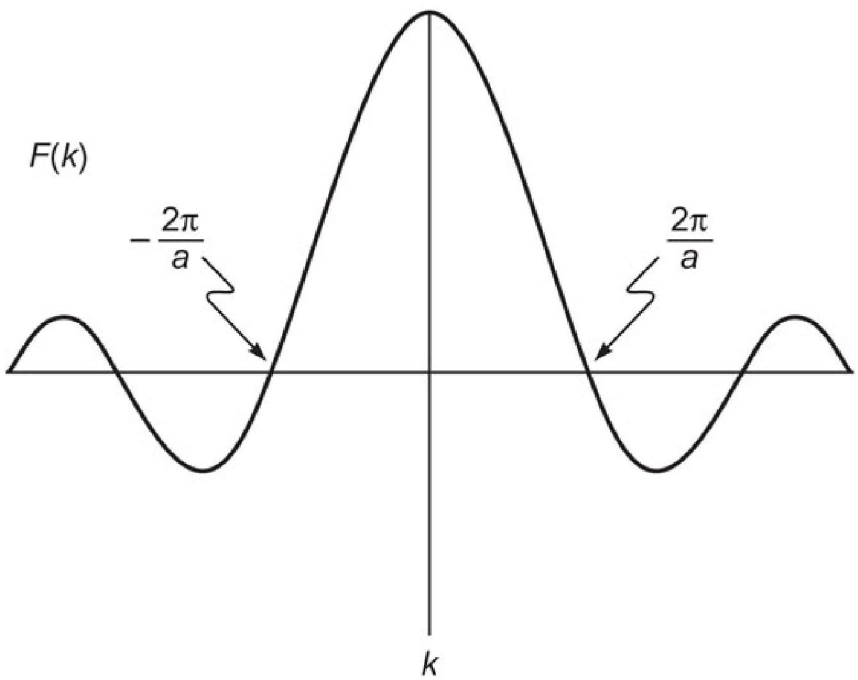
(2) The plot as shown in Fig. 9.4 in Ghatak book (P122)

The width of $f(x) : \Delta x = a$

The width of $F(k) : \Delta k \sim \dfrac{1}{a}$



(a)



(b)

2. (Adopted from Example 10.4 in Ghatak book). Given the Gaussian pulse described as

$$E(t) = E_0 e^{-t^2/\tau_0^2} e^{+i\omega_0 t}$$

(1) show that its Fourier transform is

$$E(\omega) = \frac{E_0 \tau_0}{2\sqrt{\pi}} \exp[-\frac{1}{4}(\omega - \omega_0)^2 \tau_0^2]$$

(2) make a schematic plot of E(t) and E(w) respectively (hand drawing is acceptable).

Answer:

(1)) This is a Gaussian pulse with a monotonic frequency wave
Using the Fourier Transform
with the factor $1/2\pi$

$$E(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\exp(-i\omega t)dt$$

$$E(\omega) = \frac{E_0}{2\pi}\int_{-\infty}^{+\infty} e^{-t^2/\tau_0^2} \exp(i\omega_0 t)\exp(-i\omega t)dt$$

$$E(\omega) = \frac{E_0}{2\pi}\int_{-\infty}^{+\infty} e^{-t^2/\tau_0^2} e^{[-i(\omega-\omega_0)t]}dt$$

Use the integral identify (see Appendix A of Ghatak book for the prove)

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$$

We have

$$\alpha = \frac{1}{\tau_0^2}$$

$$\beta = -i(\omega - \omega_0)$$

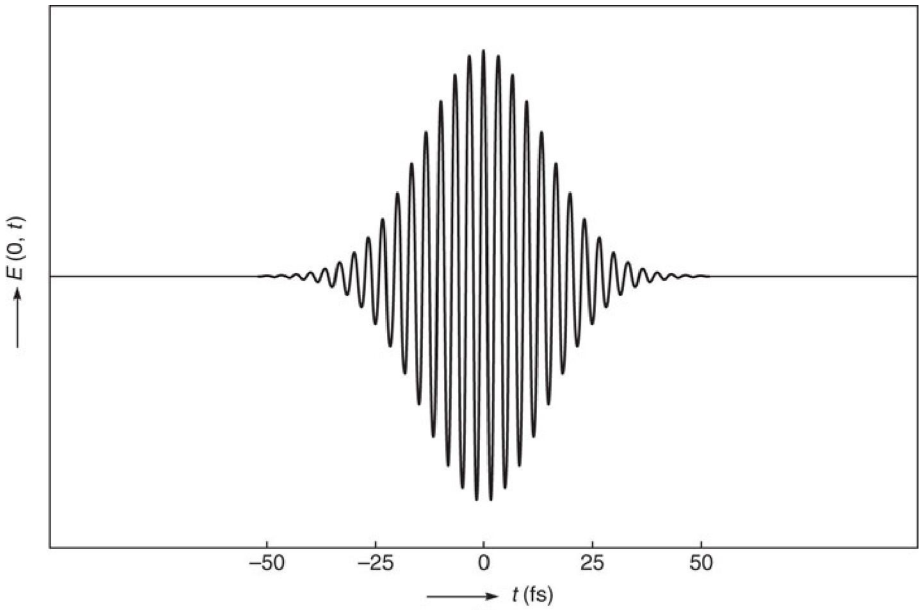
Thus

$$E(\omega) = \frac{E_0 \tau_0}{2\sqrt{\pi}} \exp[-\frac{1}{4}(\omega - \omega_0)^2 \tau_0^2]$$

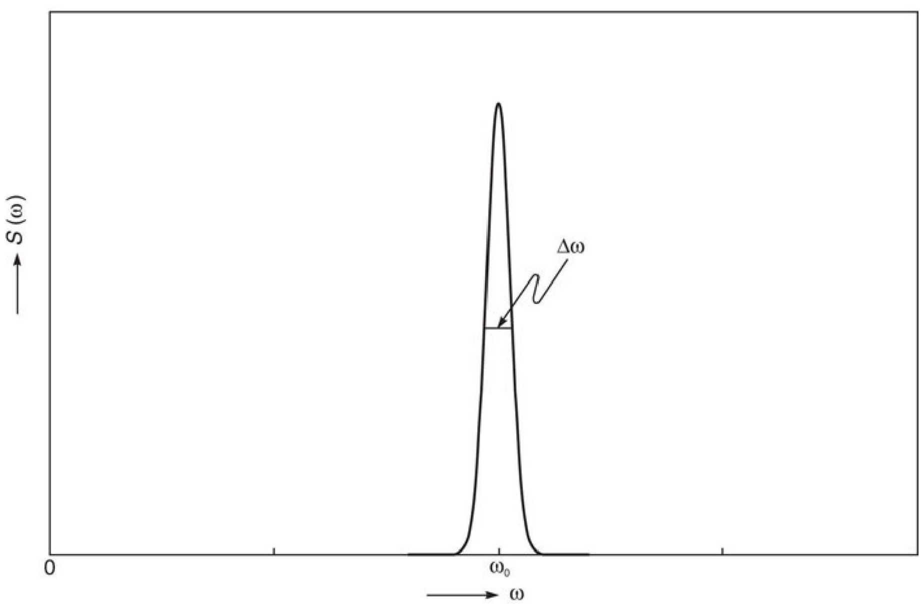
(2) The plot is shown in Fig. 10.4 in P.133 of Ghatak book

The width of $E(t): \Delta t \sim \tau_0$

The width of $E(\omega): \Delta\omega \sim \frac{1}{\tau_0}$



(a)



(b)