PHYS 306 Spring 2010
Wave Motion and Electromagnetic Radiation

Homework Assignment

HW#2
Assignment Date: Feb. 2, 2010
Due Date: Feb. 09, 2010

1. (Adopted from Problem 8.1 in Ghatak book). Consider a periodic force of the form:

$$
\begin{aligned}
F(t) &= F_0 \sin \omega t \quad &&\text{for} \quad 0 < t < T/2 \\
&= 0 &&\text{for} \quad T/2 < t < T
\end{aligned}
$$

and

$$F(t+T) = F(t)$$

where

$$\omega = 2\pi/T$$

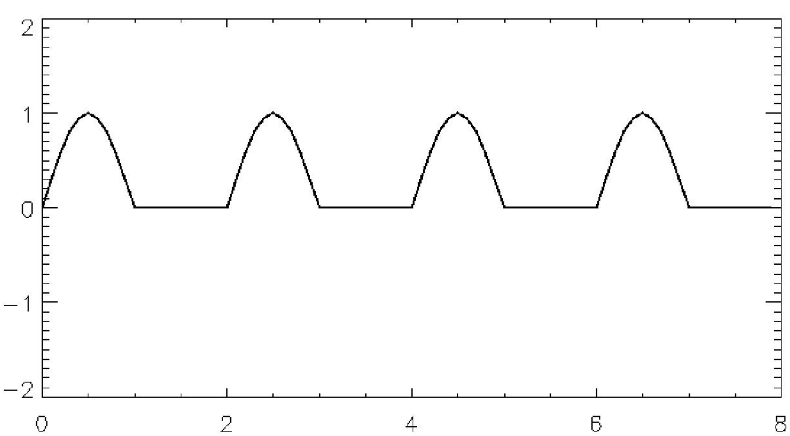
(1) Show that

$$F(t) = \frac{1}{\pi}F_0 + \frac{1}{2}F_0 \sin \omega t - \frac{2}{\pi}F_0\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \ldots\right)$$

(2) Plot the Fourier series in (1) for the summation of the components up to n=5.

Note: For the plotting, you may use your favorite software, whatever it is. One simple option is to use the online grapher at http://www.walterzorn.com/grapher/grapher_e.htm. You may use the screenshot to save the plot. Please print the plot and attach to the homework.

Answer
(1) The function F(t) is in the form of a rectified half-wave, as shown in Figure 1



In this plot, F0 = 1.0, T = 2.0 and $\omega = \pi$ are assumed.

According to Fourier Theorem

$$F(t) \;=\; \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$a_0 = \frac{2}{T}\int_0^T F(t)dt = \frac{2}{T}F_0\int_0^{+T/2} \sin \omega t\, dt = \frac{2}{T}F_0\frac{T}{2\pi}\left[-\cos\pi + 1\right] = \frac{2}{\pi}F_0$$

where we have used $\omega = 2\pi / T$.

Similarly,

$$a_n = \frac{2}{T}\int_0^T F(t)\cos(n\omega t)dt$$

$$a_n = \frac{2}{T}F_0\int_0^{+T/2} \sin \omega t \;\cos n\omega t \;dt$$

Use the trigonometric identity

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$a_n = \frac{2}{T}F_0\frac{1}{2}\int_0^{+T/2}\left[\sin(\omega t + n\omega t) + \sin(\omega t - n\omega t)\right]dt$$

$$a_n = \frac{2}{T}F_0\frac{1}{2}\int_0^{+T/2}\left[\sin\frac{2\pi}{T}(1+n)t + \sin\frac{2\pi}{T}(1-n)t\right]dt$$

$$a_n = \frac{F_0}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$a_n = \begin{cases} \dfrac{F_0}{\pi}\left[\dfrac{1}{1+n} + \dfrac{1}{1-n}\right] & \text{for } n = 0,2,4,... \\ 0 & \text{for } n = 3,5,... \end{cases}$$

a_n goes to infinite as $\mathrm{n} = 1$ in this expression.

Therefore, the case of $\mathrm{n} = 1$ has to be determined seperately

$$a_1 = \frac{2}{T}\int_0^T F(t)\cos(\omega t)dt$$

$$a_1 = \frac{2}{T}\frac{F_0}{2}\int_0^{+T/2} \sin 2\omega t\, dt = 0$$

$$a_1 = \frac{F_0}{T}\frac{T}{4\pi}(-\cos\frac{4\pi}{T}t)\Big|_0^{T/2}$$

$$a_1 = 0$$

Further,

$$b_n = \frac{2F_0}{T}\int_0^{+T/2} \sin \omega t \ \sin n\omega t \ dt = \frac{F_0}{2\pi}\left\{\frac{\sin(1-n)\omega t}{(1-n)} - \frac{\sin(1+n)\omega t}{(1+n)}\right\}_0^{T/2}$$

$$= 0 \qquad\qquad \text{for} \quad n = 2,3,4,...$$

b_1 has to be determined separately,

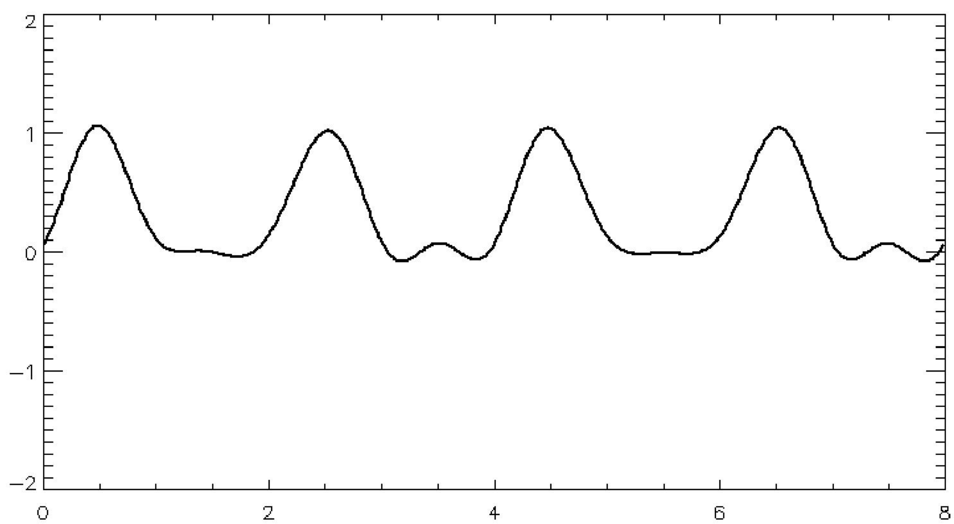
$$b_1 = \frac{2}{T} F_0 \int_0^{T/2} \sin^2 \omega t \, dt = \frac{F_0}{T}\int_0^{T/2} (1-\cos 2\omega t)\, dt = \frac{1}{2} F_0$$

Thus

$$F(t) \ = \ \frac{1}{\pi}F_0 + \frac{1}{2}F_0 \sin \omega t - \frac{2}{\pi}F_0\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + ...\right)$$

(2) The plot of F(t, n=5), the function is as described as above.

F0 = 1.0, T = 2.0 and $\omega = \pi$ are assumed.



2. (Adopted from Example 8.2 in Ghatak book, P. 113, but there is a minor difference) Consider the following periodic function

$$f(t) = \begin{cases} +A & \text{for } -\frac{T}{2} < t < 0 \\ -A & \text{for } 0 < t < \frac{T}{2} \end{cases}$$

and

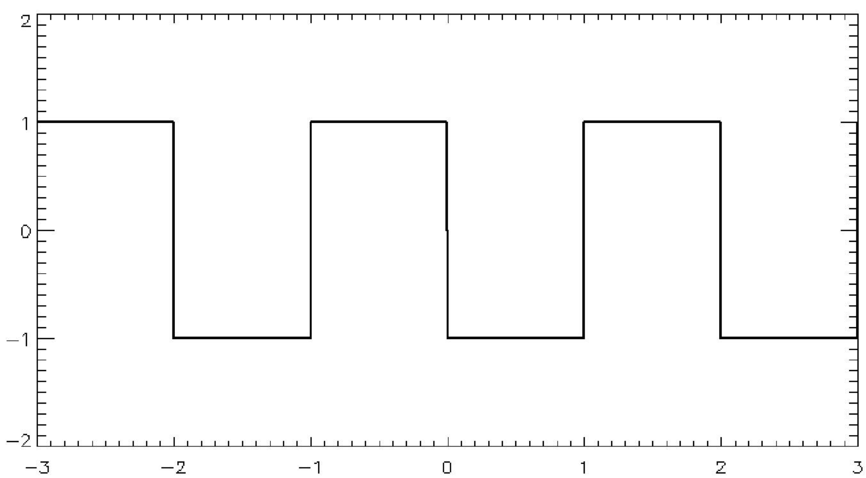
$$f(t+T) = f(t)$$

(1) Derive the Fourier series for this periodic function

(2) Plot the derived Fourier series for the summation of the components up to n=5

Answer:

(1) This is a step function. The plot is shown as, for A=1, T=2



Since it is an odd function, a_n=0. It can be approved as

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{0} A\cos(n\omega t)dt + \frac{2}{T}\int_{0}^{T/2} (-A)\cos(n\omega t)dt$$

The first term on RHS : first change the order of the integration limit

$$\int_{-T/2}^{0} A\cos(n\omega t)dt = -\int_{0}^{-T/2} A)\cos(n\omega t)dt$$

Then allow t goes to (-t)

$$-\int_{0}^{-T/2} A)\cos(n\omega t)dt = \int_{0}^{T/2} A)\cos(n\omega t)dt$$

Therefore, $a_n = 0$, including $n = 0$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(t)\,\sin n\omega t\,dt$$

The integrant is an even function (odd multiplied by odd), therefore

$$b_n = \frac{2}{T}2\int_{0}^{+T/2} f(t)\,\sin n\omega t\,dt = \frac{4}{T}\int_{0}^{+T/2}(-A)\sin n\omega t\,dt$$

$$b_n = \frac{-4A}{T}\frac{1}{n\omega}\left[-\cos(n\omega t)\right]_0^{T/2}$$

$$b_n = \frac{2A}{n\pi}[\cos(n\pi)-1]$$

$$b_n = \frac{2A}{n\pi}[(-1)^n - 1]$$

Thus

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

For n = 5

$$f(t) = \frac{-4A}{\pi}\sin(wt) - \frac{4A}{3\pi}\sin(3wt) - \frac{4A}{5\pi}\sin(5wt)$$

(2)

The plot of f(t, n=5), the function is as described as above.

A = 1.0, T = 2.0 and $\omega = \pi$ are assumed.