Nov. 11, 2010

MHD waves — A "Brief" Introduction

See: "Plasma physics for Astrophysics" by Russel M. Kulsrud
Chapter 5, P103 - P126

This topic should be covered by ~~and~~ any plasma textbooks

Start from MHD Equations

The simplest MHD motion is MHD waves

Using perturbation analysis method

$$\rho = \rho_0 + \delta\rho$$
$$P = P_0 + \delta P$$
$$\vec{B} = \vec{B}_0 + \delta\vec{B}$$
$$\vec{V} = \vec{V}_0 + \delta\vec{V}$$

Consider locally homogeneous medium

~~$\rho_0 = 0$~~, ρ_0, P_0, $\vec{B}_0$ are constants

and $\boxed{\vec{V}_0 = 0}$

Further consider perturbation is a linear wave

$$\delta\rho = \rho_1 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ ρ_1: wave amplitude

or $\rho_1 \cos\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right)$ $\vec{k}$: wave vector

$|\vec{k}| = \frac{2\pi}{\lambda}$: λ: wavelength

$$\delta P = P_1 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\delta\vec{B} = B_1 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ ω: wave frequency
$$\delta\vec{V} = V_1 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ $\omega = \frac{2\pi}{T}$: T: period

$$\frac{\partial\rho}{\partial t} = \frac{\partial\delta\rho}{\partial t} = \rho_1(-i\omega)e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$\nabla\rho = \rho_1(i\vec{k})e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Use MHD equations (including Maxwell equations), one could find solutions of : by dispersion relation

① Alfven wave mode (intermediate wave)

$$\frac{\omega^2}{K^2} = V_A^2 \cos^2\theta$$

θ : angle between $\hat{K}$ and $\hat{B}$

$\hat{K}$: direction of wave propagation

$$V_A = \pm \frac{\omega}{K} \cos\theta$$

$$V_A = \frac{B_0}{\sqrt{\mu \rho_0}}$$: speed of Alfven wave

② Fast mode MHD waves
Slow mode MHD waves

$$\frac{\omega^2}{K^2} = \frac{V_A^2 + C_s^2}{2} \pm \frac{1}{2}\sqrt{(V_A^2 - C_s^2)^2 + 4 C_s^2 V_A^2 \sin^2\theta}$$

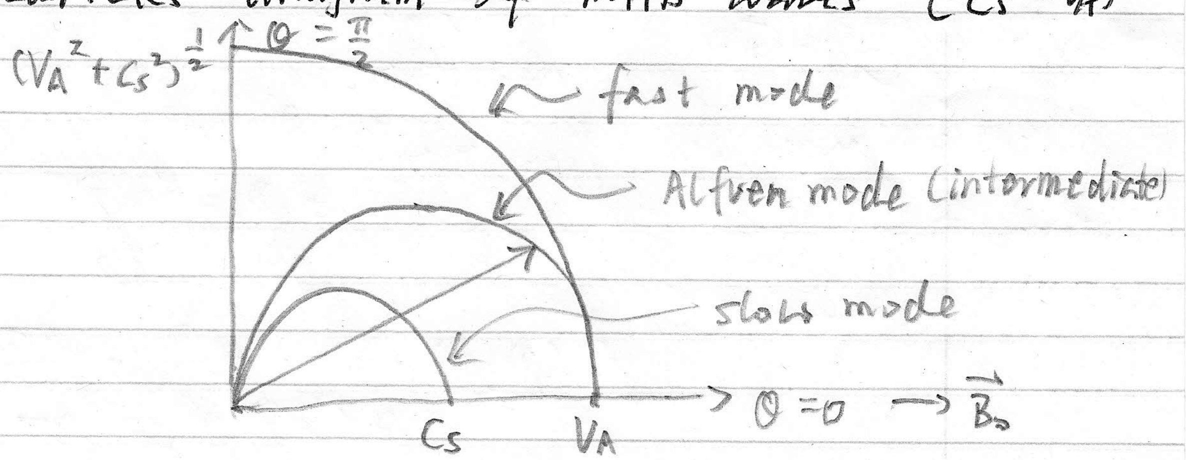
$$C_s = \sqrt{\frac{\gamma P_0}{\rho_0}}$$: sound speed
Acoustic wave speed

"+" : fast mode ; "−" : slow mode

Further, wave has phase speed $V_p = \frac{\omega}{k}$
and group speed $V_g = \frac{d\omega}{dk}$

For linear wave, $V_g = V_{ph} = \frac{\omega}{K}$

* Friedricks diagram of MHD waves ($C_s < V_A$)

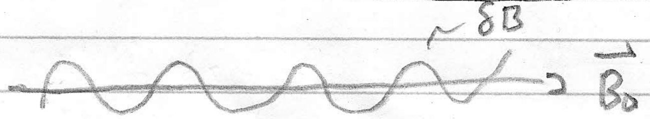


Intermediate mode (Alfven wave)

$$V_g = V_p = \frac{\omega}{k} = \pm V_A \cos\theta$$

$\vec{k} \parallel \vec{B}_0$: $V_{ph} = \pm V_A$: largest speed when wave propagate along magnetic field

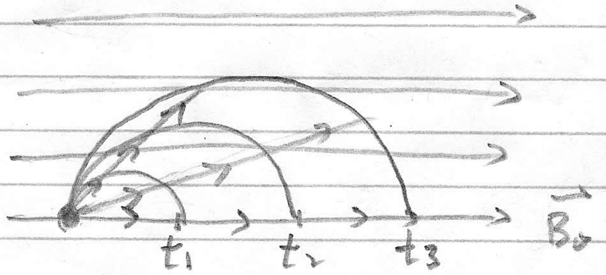
$\delta\vec{B} \perp \vec{B}_0$: (transverse wave by magnetic tension force)



$\delta\rho = 0$

$\vec{k} \perp \vec{B}_0$: $V_{ph} = 0$

Alfven wave speed is zero, when wave direction is perpendicular to $\vec{B}_0$



Alfven wave front

slow mode

parallel propagation $\vec{k} \parallel \vec{B}_0$: $V_{ph} = C_s$ $\quad C_s < V_A$

$V_{ph} = V_A \quad V_A < C_s$

Alfven mode : transverse motion only

Sound mode : parallel motion only.

Sound mode and Alfven mode do not interact

perpendicular propagation : $V_{ph} = 0$

δP and δB act in different direction

fast mode

* parallel propagation:

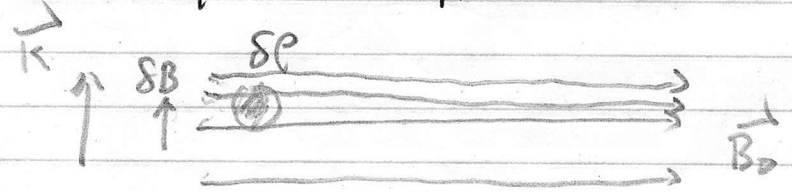
$$V_{ph} = V_A \quad \text{if} \quad V_A > C_s$$

Again, when $\hat{k} \parallel \hat{B}$: sound mode and Alfven mode decouple: or not interact with each other.

* However, in perpendicular propagation.

$$\vec{k} \perp \vec{B}.$$

perpendicular motion cause magnetic field become denser or compressed, due to frozen-in effect

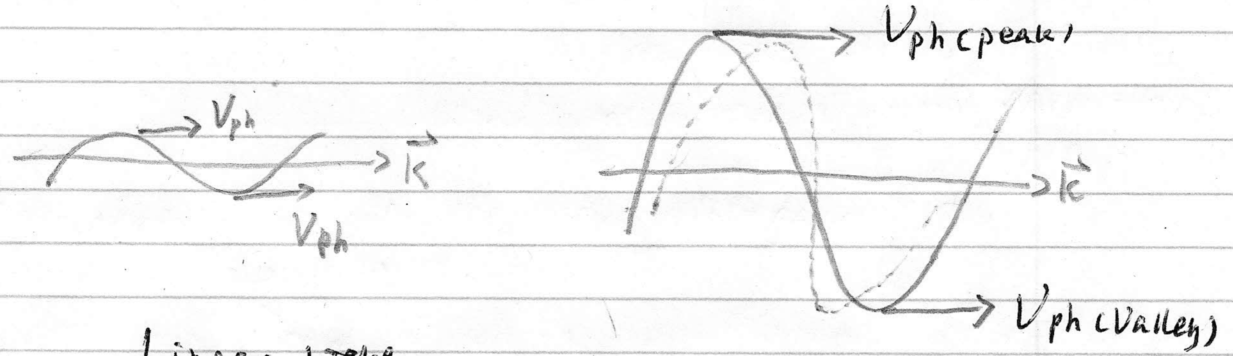


magnetic pressure force and thermal pressure force re-inforce each other; causing fast-mode wave motion.

$$V_{ph} = (V_A^2 + C_s^2)^{\frac{1}{2}}$$

Shocks

If a wave has large amplitude, it will steepen into a shock, because different parts of a wave have different velocity

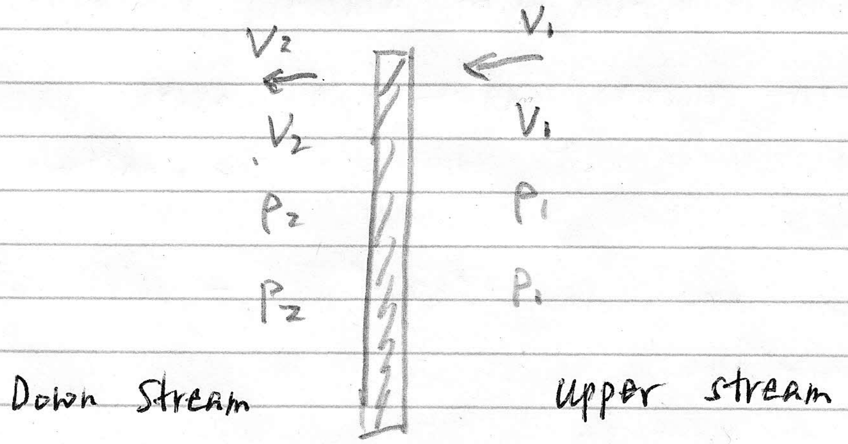


Linear wave
$\delta\rho \ll \rho_0$

non-linear wave
$\delta\rho \sim \rho_0$
$V_{ph}(\text{peak}) > V_{ph}(\text{valley})$

Ref: "plasma physics for Astrophysics" by Russell M. Kulsrud. Chap. 6

Ref. Priest & Forbes book: CHap 1-5.



Shock front (moving at velocity

In shock frame, there are two homogeneous mediums.

upper stream: V_1, ρ_1, P_1
lower stream: V_2, ρ_2, P_2

* For shock wave of sound mode

always $\rho_2 > \rho_1$ compression

$P_2 > P_1$ $\Uparrow$

$V_2 < V_1$ slow down

Solve fluid equations become $\boxed{\text{jump conditions}}$

Consider steady state

1. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \Rightarrow \nabla \cdot \rho \vec{V} = 0$

$\Rightarrow \rho_1 V_1 = \rho_2 V_2$: mass conservation

2. $\cancel{\rho \frac{\partial \vec{V}}{\partial t}} + \rho (\vec{V} \cdot \nabla) \vec{V} = - \nabla P$

$\Rightarrow \rho_1 V_1^2 + P_1 = \rho_2 V_2^2 + P_2$

: momentum conservation

3. $(\frac{1}{2}\rho_1 V_1^2 + P_1 + \rho_1 e_1) V_1 = (\frac{1}{2}\rho_2 V_2^2 + P_2 + \rho_2 e_2) V_2$

: energy conservation

Introduce $X = \frac{\rho_2}{\rho_1}$: density ratio, or compression ratio

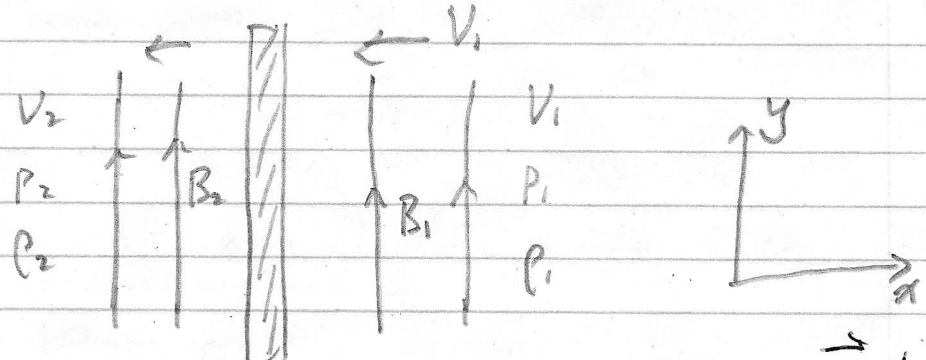
$X = \frac{V_2}{V_2}$

shock Mach number $M_1 = \frac{V_1}{C_{s1}}$

V_1 - typically the shock speed for a moving shock

~~* For MHD shocks~~

* For MHD shocks : (1) Having $\vec{B}$, (2) direction of $\vec{B}$



Perpedicular MHD shock : $\vec{V_1} \perp \vec{B_1}$

* Perpendicular shock $\vec{V_1} \perp \vec{B_1}$

$$\nabla \cdot \vec{B} = 0, \Rightarrow \oint_s \vec{B} \cdot d\vec{s} = 0 \Rightarrow B_{1x} = B_{2x}$$

$\Rightarrow \vec{B}$ continuous along the normal direction of shock front

However $\nabla \times \vec{B} = \mu_0 \vec{j}$

$$\oint_l \vec{B} \cdot d\vec{l} = \mu_0 \vec{j} \Rightarrow B_{1y} - B_{2y} = \mu_0 j$$

If $\vec{j} \neq 0$, $B_{1y} = B_{2y}$

$\Rightarrow \vec{B}$ jumps along the tangential direction of the shock front

(1) $\rho_1 V_1 = \rho_2 V_2$

(2) $\rho_1 V_1^2 + P_1 + \frac{B_1^3}{2\mu_0} = \rho_2 V_2^2 + P_2 + \frac{B_2^2}{2\mu_0}$

(3) ---- energy

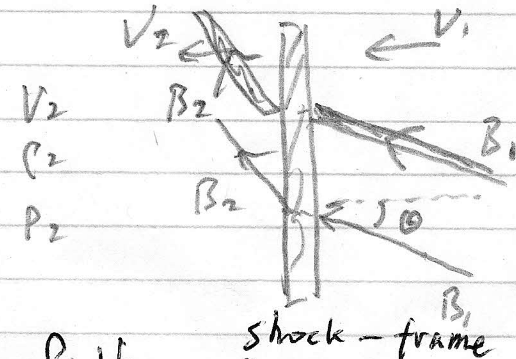
(4) $B_1 V_1 = B_2 V_2 = E$ = constant in 2-D

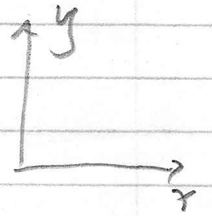
$$X = \frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{B_2}{B_1}$$

$$M_1 = \frac{V_1}{C_{s1}}$$

Oblique MHD shocks $\vec{B} \not\perp \vec{V}$

$\vec{B}$ is oblique to $\vec{V}$ and shock front





shock – frame

(1) $\rho_1 V_{1x} = \rho_2 V_{2x}$

(2) $$P_1 + \rho_1 V_{1x}^2 + \frac{B_1^2}{2\mu} - \frac{B_{1x}^2}{\mu} = P_2 + \rho_2 V_{2x}^2 + \frac{B_2^2}{2\mu} - \frac{B_{2x}^2}{\mu}$$

(3) $$\rho_1 V_{1x} V_{1y} - \frac{B_{1x} B_{1y}}{\mu} = \rho_2 V_{2x} V_{2y} - \frac{B_{2x} B_{2y}}{\mu}$$

(3) energy ...

(4) $B_{1x} = B_{2x}$

(5) $V_{1x} B_{1y} - V_{1y} B_{1x} = V_{2x} B_{2y} - V_{2y} B_{2x}$

If choose $V_{1y} = V_{1x} \dfrac{B_{1y}}{B_{1x}}$: an arbitrary value

Jump conditions can be simplified, when $\boxed{\vec{V} \parallel \vec{B}}$

$$X = \frac{\rho_2}{\rho_1}, \quad C_{s1} = \frac{\gamma P_1}{\rho_1}, \quad V_{A1} = \frac{B_1^2}{\mu_0 \rho_1}, \quad \cos\theta = \frac{V_{1x}}{V_1}$$

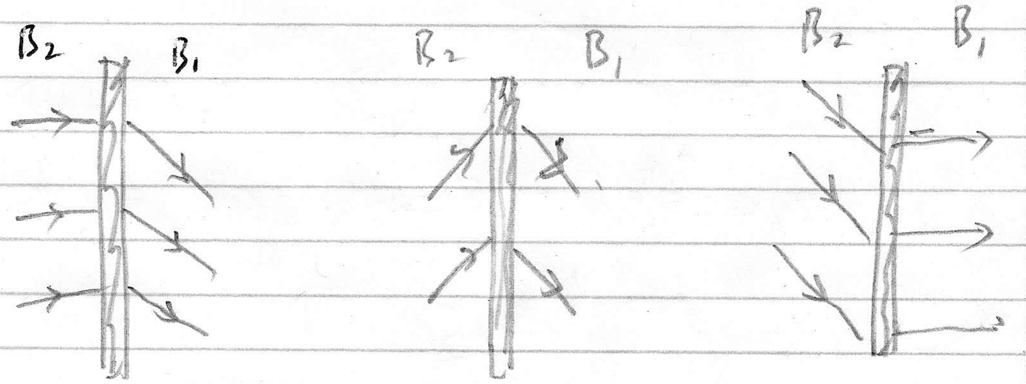
$$\frac{V_{2x}}{V_{1x}} = \frac{1}{X}, \quad \frac{V_{2y}}{V_{1y}} = \frac{V_1^2 - V_{A1}^2}{V_1^2 + X V_{A1}^2}, \quad \frac{B_{2x}}{B_{1x}} = 1,$$

$$\frac{B_{2y}}{B_{1y}} = \frac{(V_1^2 - V_{A1}^2) X}{V_1^2 - X V_{A1}^2}$$

(continued)

X has three kinds of value, forming

1. slow-mode shock
2. Alfven wave
3. fast-mode shock



slow mode.	Alfven mode	fast mode
$B_2 < B_1$	$B_2 = B_1$	$B_2 > B_1$
$B_{2y} = 0$	$B_{2y} = -B_{1y}$	$B_{2y} > B_{1y} = 0$
$B_{1y} \neq 0$	$B_{2x} = B_{1x}$	$B_{2x} = B_{1x}$
$B_{2x} = B_{1x}$	$X = \frac{\rho_2}{\rho_1} = 1$	$X = \frac{\rho_2}{\rho_1} > 1$
$X = \frac{\rho_2}{\rho_1} > 1$	$(\rho_2 = \rho_1)$	

slow mode: B_y goes to zero, $\boxed{\text{switch-off shock}}$

fast mode: B_y gained from zero: $\boxed{\text{switch-on shock}}$

* slow-mode shock: responsible for Petschek mechanism

Since $\vec{V} \parallel \vec{B}$: $V_{1x} = \frac{B_{1x}}{\sqrt{\mu\rho}} \Rightarrow$ shock front propagates at the speed based on $\vec{B}$ normal.

$V_{1y} = \frac{B_{1y}}{\sqrt{\mu\rho}} \Rightarrow$ frame of reference along y moves at the ~~speed~~ of $\vec{B}$ parallel