Oct. 14, 2010

Steady state of magnetic field

So far, we've studied only the static $\vec{B}$:

$\vec{v} = 0$

$\vec{B}$: potential, force-free, null points

current sheet

Now, we study the steady state

$\vec{v} \neq 0$, flow exists

$\frac{\partial}{\partial t} = 0$, e.g. $\frac{\partial V}{\partial t} = 0$, $\frac{\partial \vec{B}}{\partial t} = 0$ – no time variation

Magnetic Induction Equation CH 1.2.2

Frozen Flux CH 1.4

Magnetic Annihilation CH 3.1, CH 3.2

— 1-D magnetic field

Magnetic Induction Equation (CH 1.2.2)
– Magnetic Reynolds number

$\vec{B}$ Induction equation originates from Generalized Ohm's Law

$$\vec{j} = \sigma \vec{E}' ,$$

σ : plasma conductivity

$\vec{E}'$: Electric field in plasma moving frame $\vec{V}$

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

$\vec{E}$: $\vec{E}$ field in the frame of laboratory
rest frame

$\vec{v} \times \vec{B}$: $\vec{E}$ field induced from the advection motion
or Lorentz transformation term

$$\Rightarrow \vec{j} = \sigma (\vec{E} + \vec{v} \times \vec{B})$$

$$\Rightarrow \vec{E} = \frac{\vec{j}}{\sigma} - \vec{v} \times \vec{B}$$

From Faraday's Law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{\partial \vec{B}}{\partial t} = -(\nabla \times \vec{E})$$

$$\nabla \times \vec{E} = \frac{1}{\sigma} \nabla \times \vec{J} - \nabla \times (\vec{v} \times \vec{B})$$

$$\nabla \times \vec{E} = \frac{1}{\sigma \mu_0} [\nabla \times (\nabla \times \vec{B})] - \nabla \times (\vec{v} \times \vec{B})$$

$$\cancel{\nabla \times \nabla \times \vec{B} = (\nabla \cdot \nabla) \vec{B}}$$

From vector identity of

$$\nabla \times (\nabla \times \vec{B}) = \cancel{\nabla (\nabla \cdot \vec{B})} - \nabla^2 \vec{B}$$

$$\nabla^2 \vec{B} = (\nabla \cdot \nabla) \vec{B}$$

Therefore

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma \mu_0} \nabla^2 \vec{B}$$

or $$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 B}$$

magnetic induction equation

η : magnetic diffusivity

$$\eta = \frac{1}{\sigma \mu_0} = \frac{\eta_e}{\mu_0}$$

η_e : electric ~~conductivity~~ resistivity

σ : electric conductivity

$\eta_e = \frac{1}{\sigma}$

convection term : $\nabla \times (\vec{v} \times \vec{B})$

This is how magnetic field generated through solar dynamo, or differential motion of the Sun

diffusion term : $\eta \nabla^2 \vec{B}$:

This is how magnetic field dissipated

1. through resistivity, but η small
2. through ~~reconnection~~ current sheet, $\nabla^2 \vec{B}$ large

Magnetic Reynolds number

$$R_m = \frac{\text{convection term}}{\text{Diffusion term}} = \frac{\nabla \times (\vec{v} \times \vec{B})}{\eta \nabla^2 B}$$

Using dimensional analysis and characteristic sclales

$\nabla = \frac{1}{L_0}$ $\quad$ L_0: characteric length

$\vec{v} \rightarrow V_0$ $\quad$ V_0: characteristic velocity

$$\nabla \times (\vec{v} \times \vec{B}) = \frac{1}{L_0}(V_0 B_0)$$

$$\nabla^2 B = \frac{1}{L_0^2} B_0$$

$$\boxed{R_m = \frac{L_0 V_0}{\eta}}$$

For fluid, $R = \frac{L_0 V_0}{\nu}$

where ν is viscousity

From Spitzer (1962)

$$\eta = \frac{c^2 e^2 m_e^{\frac{1}{2}}}{3(2\pi)^{3/2} \varepsilon_0} \ln\Lambda \, (K_B T_e)^{-3/2} \quad \text{--- (1.13)}$$

$$= 1.05 \times 10^8 \, T_e^{-3/2} \ln\Lambda \quad m^2 s^{-1}$$

$\ln\Lambda$: the Coulomb logarithm

$$\ln\Lambda = \begin{cases} 16.3 + \frac{3}{2}\ln T - \frac{1}{2}\ln n & T < 4.2 \times 10^5 K \\ 22.8 + \ln T - \frac{1}{2}\ln n & T > 4.2 \times 10^5 K \end{cases}$$

Exp. Active region corona

$T \sim 10^6 K$, $n = 10^{15} m^{-3}$, $L_0 \sim 10^5 m$, $V_0 = 10^4 m s^{-1}$

$\Rightarrow R_m = 10^9$

$\eta = 1 \, m^2 s^{-1}$.

convection terms dominates. diffusion term negligible

Diffusion time scale τ_d

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B}$$

$$\frac{B_0}{\tau_d} = \eta \frac{B_0}{L_0^2}$$

$$\tau_d = \frac{L_0^2}{\eta}$$

For a sunspot, $L_0 = 10^5$ m, $\eta = 1\ m^2 s^{-1}$
magnetic loop, magnetic element

$$\tau_d = 10^{10} \text{ sec} = 300 \text{ years}$$

Sunspot persists; its dissappearance depends on the surface turbulent motion

Frozen Flux (CH 1.4)

— Also called "Frozen-in" Effect

In a fluid with large magnetic Reynold number,

$$R_m \gg 1,$$

or ideal fluid, that is $\sigma \to \infty$, $\eta \to 0$, $\Rightarrow R_m \gg 1$,

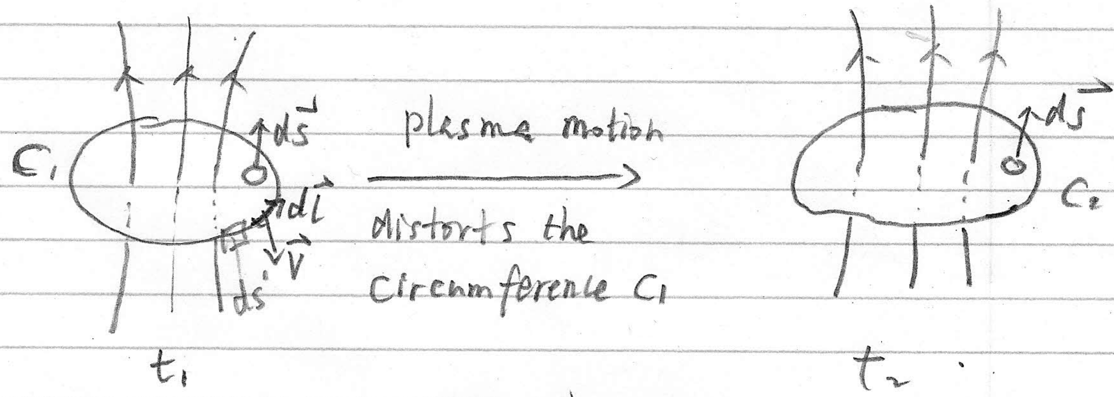
$\Rightarrow$ the field lines move as though they are "frozen" with the moving fluid

Magnetic Induction Eq.

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B})$$

There is no diffusion of magnetic field ($\eta = 0$), but there is bulk motion of plama, that changes $\vec{B}$

The bulk motion of plasma drags the magnetic field line.



magnetic flux $\phi = \int_S \vec{B} \cdot d\vec{s}$

the rate of change of ϕ through C is

$$\frac{d}{dt}\int \vec{B}\cdot d\vec{s} = \underbrace{\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}}_{\text{local charge}} + \underbrace{\oint_{C_1} \vec{B}\cdot(\vec{V}\times d\vec{l})}_{\text{boundary charge of flux}}$$

charge of boundary

charge of area ds'

$\vec{B} \cdot \vec{V} \times d\vec{l} = -\vec{V} \times \vec{B} \cdot d\vec{l}$ ← vector identity

Stoke's theorem $\int_C \vec{V} \cdot d\vec{l} = \int_S (\nabla \times \vec{V}) \cdot d\vec{s}$

$$\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \cancel{+} - \int_C (\vec{V} \times \vec{B}) \cdot d\vec{l}$$

$$= \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} - \int_S \nabla \times (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

$$= \int_S \left[\underline{\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B})} \right] \cdot d\vec{s} = 0$$

⇒ No matter how the boundary deforms, total magnetic flux ~~through~~ within the boundary is the same.

⇒ merely magnetic flux conservation

$$\boxed{\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s} = 0}$$

magnetic Annihilation CH 3

Diffusion (CH 3.1.1)

$R_m >> 1$, magnetic frozen-in with flow field

$R_m << 1$, magnetic diffusion dominates the change of $\vec{B}$

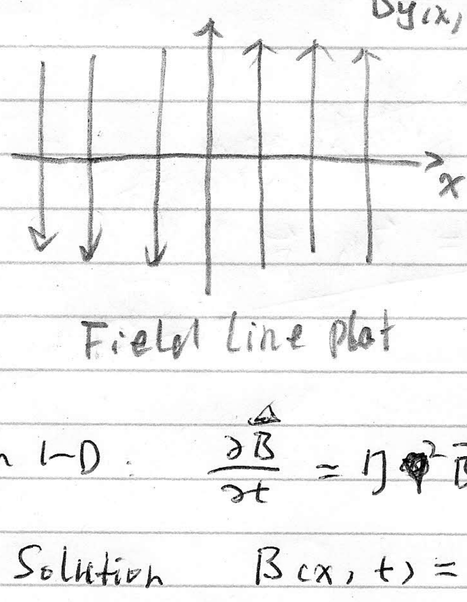
$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B}$$

* Considering one-dimensional field : 1-D

$\vec{B}$ only has $\hat{y}$ component, infinite long ~~to~~ along y

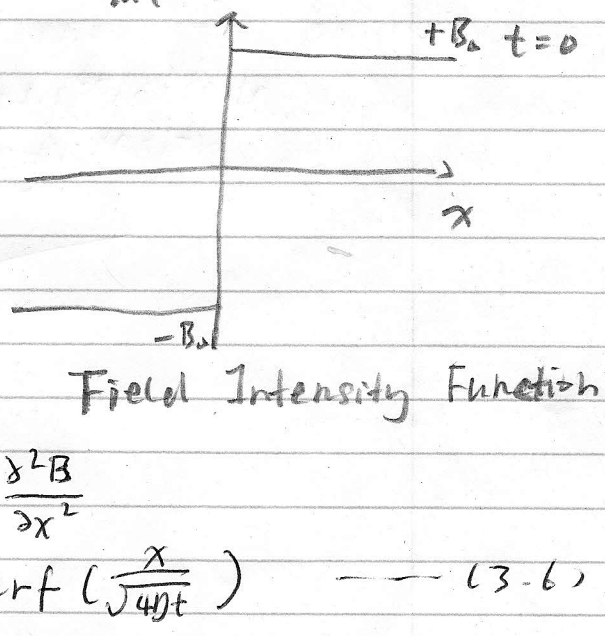
$\vec{B}$ only changes along X

At $t = 0$



Field Line plot

At $t = 0$

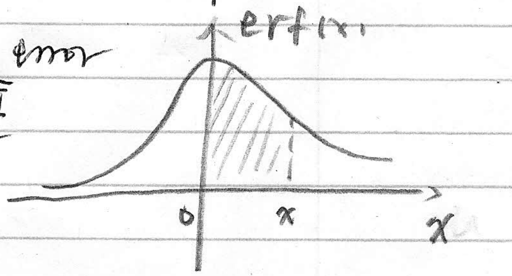


Field Intensity Function

In 1-D : $\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} = \eta \frac{\partial^2 B}{\partial x^2}$

Solution $B(x, t) = \frac{2 B_0}{\sqrt{\pi}} \, erf\left(\frac{x}{\sqrt{4\eta t}}\right)$ —— (3.6)

$erf(x) = \int_0^x \underbrace{e^{-u^2}}_{\text{Gaussian for random error}} du$: the error function



$x \to \infty \Rightarrow erf(x) = \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$

(9)

prove the solution is correct

$$\frac{\partial B}{\partial t} = \frac{2B_0}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^{\frac{x}{\sqrt{4\eta t}}} e^{-u^2} du$$

Use the rule: $\frac{d}{dt}\int_0^{b(t)} f(x)\,dx = \underbrace{\frac{db(t)}{dt}} \cdot \underbrace{f[b(t)]}$

$$b(t) = \frac{x}{\sqrt{4\eta t}} \qquad \frac{db(t)}{dt} = \frac{x}{\sqrt{4\eta}} \cdot \left(-\frac{1}{2}\right) \frac{1}{t}$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{2B_0}{\sqrt{\pi}} \cdot \frac{x}{\sqrt{4\eta}} \left(-\frac{1}{2}\right) \frac{1}{t^{\frac{3}{2}}} e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2} = -\frac{B_0}{\sqrt{\pi}} \frac{x}{\sqrt{4\eta}} \frac{1}{t^{3/2}} e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$$

Similarly, $\frac{\partial B}{\partial x} = \frac{2B_0}{\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^{\frac{x}{\sqrt{4\eta t}}} e^{-u^2} du = \frac{2B_0}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{4\eta t}} e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$

$$\frac{\partial^2 B}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial B}{\partial x}\right) = \frac{2B_0}{\sqrt{\pi}} \frac{1}{\sqrt{4\eta t}} \cdot (-2x) \frac{1}{\sqrt{4\eta t}} \frac{1}{\sqrt{4\eta t}} e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$$

$$\frac{\partial^2 \vec{B}}{\partial x^2} = -\frac{B_0}{\sqrt{\pi}} \frac{4x}{(4\eta t)^{3/2}} e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$$

$$\Rightarrow \frac{\partial B}{\partial t} = \eta \frac{\partial^2 \vec{B}}{\partial x^2}$$

$$B(x,t) = \frac{2B_0}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4\eta t}}} e^{-u^2} du$$

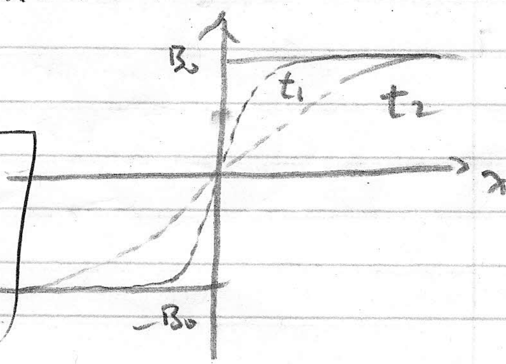
$t=0$, $x=-0$, $B(x,t) = \frac{2B_0}{\sqrt{\pi}} \int_0^{-\infty} e^{-u^2} du = -B_0$

$x=+0$, $B(x,t) = \frac{2B_0}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du = +B_0$

$t = t_1$, $B(x,t) \downarrow$,

$x \downarrow$, $B(x,t) \downarrow$.

$\boxed{\text{Magnetic field diffuse toward the center at } x=0\text{, and annihilate there}}$



* Current $\vec{j}$

$$\vec{j} = \frac{1}{\mu}\nabla\times\vec{B} = \frac{1}{\mu}\frac{\partial B(x,t)}{\partial x} = \frac{1}{\mu}\frac{2B_0}{\sqrt{\pi}}\frac{1}{\sqrt{4\eta t}}e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$$

total $\vec{J} = \int_{-\infty}^{+\infty} j(x,t)\,dx$

$$J = \int_{-\infty}^{+\infty}\frac{1}{\mu}\frac{\partial B}{\partial x}dx = \frac{1}{\mu}\int_{-\infty}^{+\infty}\frac{2B_0}{\sqrt{\pi}}\frac{1}{\sqrt{4\eta t}}e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}dx$$

$$J = \frac{1}{\mu}\frac{2B_0}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-u^2}du = \frac{2B_0}{\mu}$$

Total current is conserved with time

But current spreads out with time, $\vec{j}(x,t)$

$t=0$, $j(x,t=0)$ singular point at $x=0$

$t=t_1$ $\quad j(x,t=t_1) = \frac{1}{\mu}\frac{2B_0}{\sqrt{\pi}}\frac{1}{\sqrt{4\eta t}}e^{-\left(\frac{x}{\sqrt{4\eta t}}\right)^2}$

$t\uparrow$, $j\downarrow$

$x\uparrow$, $j\downarrow$