Oct. 7, 2010

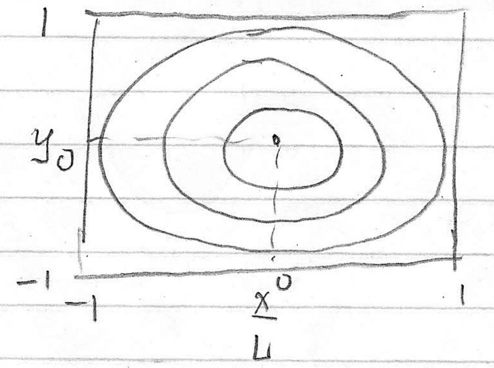
current sheet

null point solution CH 1.3

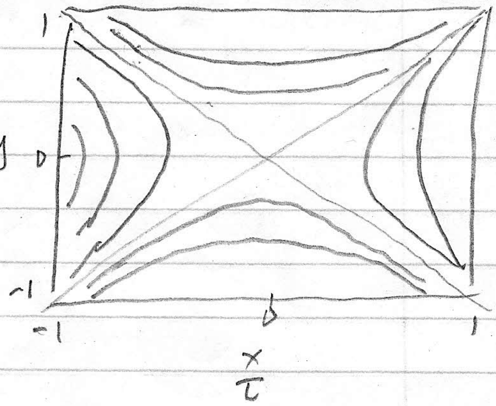
X-point collapse CH 2.1

Current sheet in Potential Fields CH 8.2

Two-dimensional null point CH 1.3.1, also Asch CH 5.6.1



O-type null point



X-type null point

null point $B_x(x=0, y=0) = 0$

$B_y(x=0, y=0) = 0$

$$B_x(x, y) = \underbrace{B_x(x=0)}_{=0} + \left.\frac{\partial B_x}{\partial x}\right|_{x=0} x + \left.\frac{\partial B_x}{\partial y}\right|_{y=0} y + \cdots$$

Taylor expansion

Keep the first order

$B_x(x, y) = ax + by$ a, b, c, d are constants

$B_y(x, y) = cx + dy$,

* Use vector potential $\vec{A}$, convenient in 2D

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \nabla \times \vec{A} = \begin{pmatrix} i & \partial/\partial x & A_x \\ j & \partial/\partial y & A_y \\ k & \partial/\partial z & A_z \end{pmatrix}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Since $B_z \equiv 0$. $A_x = 0$, $A_y = 0$

$\Rightarrow \vec{A}$ only has component in $\hat{z}$ direction, reduce to a scalar function A

$$\Rightarrow \begin{cases} B_x = \dfrac{\partial A}{\partial y} & = ax + by \\ B_y = -\dfrac{\partial A}{\partial x} & = cx + dy \end{cases}$$

The general solution of A

$$A_{(x,y)} = ax^2 + bxy + cy^2, \quad a, b, c: \text{arbitrary constants}$$

considering symmetry to X, and y, $b = 0$

A symmetric solution is

$$A(x,y) = \frac{B_0}{2L_0}(y^2 - \bar{\alpha}^2 x^2)$$

$$\Rightarrow \begin{cases} B_x = \dfrac{B_0}{L_0} y \\ B_y = \dfrac{B_0}{L_0} \bar{\alpha}^2 x \end{cases}$$

At the boundary $x = L_0$ or $y = L_0$ $\Rightarrow$ $B_x = B_0$, $B_y = \bar{\alpha}^2 B_0$

Further, the field line of 2-D magnetic field

$$\frac{dx}{B_x} = \frac{dy}{B_y} \Rightarrow \frac{dx}{y} = \frac{dy}{\bar{\alpha}^2 x}$$

$$\Rightarrow y\,dy = \bar{\alpha}^2 x\,dx \Rightarrow \underbrace{y^2 - \bar{\alpha}^2 x^2}_{\text{the same as } A} = A$$

Thus, $A(x,y)$ is also the function of the field line

$A(x,y)$ is also called magnetic flux function in PF

$A(x,y) = y^2 - \bar{\alpha}^2 x^2$

$\bar{\alpha}^2 < 0$, $y^2 + ax^2 = A$, elliptical shape $\Rightarrow$ O-type null

$\bar{\alpha}^2 = -1$ $y^2 + x^2 = A$ circular shape

$\bar{\alpha}^2 > 0$ $y^2 - ax^2 = A$. hyperbolic shape $\Rightarrow$ X-type

The seperatrix curve, or the limiting field line

$A = 0$ $y^2 - \bar{\alpha}^2 x^2 = 0$

$y^2 = \bar{\alpha}^2 x^2$

$y = \pm \bar{\alpha} x$. two straight lines cross the corner

* What about the current?

$\vec{j} = \frac{1}{\mu_0} \nabla \times \vec{B} = \frac{1}{\mu_0} (\nabla \times (\nabla \times \vec{A})) = \frac{1}{\mu_0} (\nabla (\nabla \cdot \vec{A})) - \frac{1}{\mu_0} \nabla^2 \vec{A}$

$\nabla \cdot \vec{A} = 0$

$\vec{j} = -\frac{1}{\mu_0} \nabla^2 \vec{A}_z$

$\vec{j}$ has only $\hat{z}$ component $j_x = 0$, $j_y = 0$, $j = j_z$

$\vec{j} = -\frac{1}{\mu_0} \left(\frac{d^2}{dx^2} A + \frac{d^2}{dy^2} A + \cancel{\frac{d^2}{dz^2} A} \right)$

$\vec{j} = -\left(\frac{B_0}{\mu L_0} \right) (1 - \bar{\alpha}^2) \hat{z}$ —— (1-32)

If B_x, B_y is linear, $\vec{j}$ is constant every where

O-type $\bar{\alpha}^2 < 0$ $j \neq 0$

$\vec{j} \times \vec{B} \neq 0$ $\Rightarrow$ unstable

X-type, only if $\bar{\alpha} = 1$, $j = 0$. potential field

the two seperatrix line is 90°

if $\bar{\alpha} \neq 1$, $\vec{j} \neq 0$, $\vec{j} \times \vec{B} \neq 0 \Rightarrow$ unstable

X-point Collapse — Formation of Current Sheet

CH 2.1

X-point is intrinsically unstable

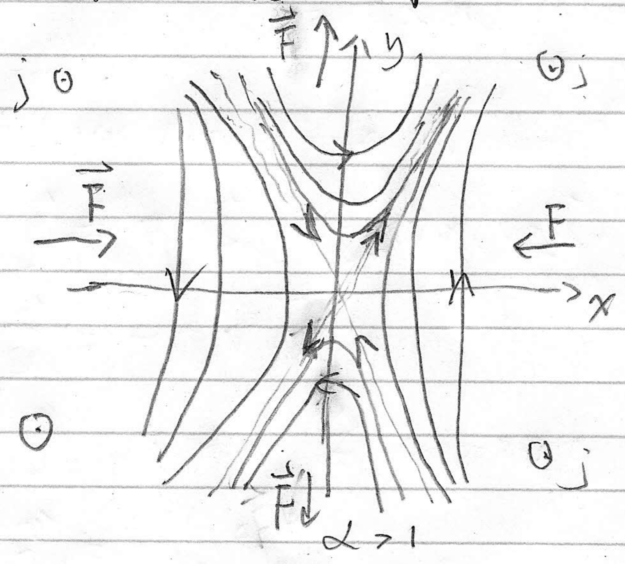
Only potential X-point is $\bar{\alpha}^2 = 1$

$$\begin{cases} B_y = x \\ B_x = y \end{cases}$$

$$A(x,y) = y^2 - x^2$$

Field lines are rectangular hyperbola $y^2 - x^2 = c$

If the field lines are perturbed, e.g., $\bar{\alpha}$ slighty > 1



$$\begin{cases} B_x = y \\ B_y = \bar{\alpha}^2 x \end{cases}$$

Fig 2.1 (b)

$$j_z = \frac{\bar{\alpha}^2 - 1}{\mu}$$

$$\vec{F} = \vec{j} \times \vec{B} = \begin{vmatrix} i & 0 & \bar{\alpha}^2 y \\ j & 0 & x \\ k & \frac{\bar{\alpha}^2-1}{\mu} & 0 \end{vmatrix} = \begin{vmatrix} -\frac{(\bar{\alpha}^2-1)\bar{\alpha}^2 x}{\mu} \\ \frac{(\alpha^2-1)y}{\mu} \\ 0 \end{vmatrix}$$

On x-axis $y = 0$, $\vec{F}$ = positive x direction for $x < 0$

$\vec{F}$ = negative x direction for $x > 0$

Along the X-axis, plasma is further squeezed toward the center

Along the Y-axis $\vec{F}$ = positive for $y > 0$

$\vec{F}$: negative for $y < 0$

plasma is moving away from the center

* Because plasma moves away from the center along Y-axis, after a series of quasi-static evolution, a current sheet is expected to form.

* the original null point bifurcate into two points. at the two ends of the current sheet

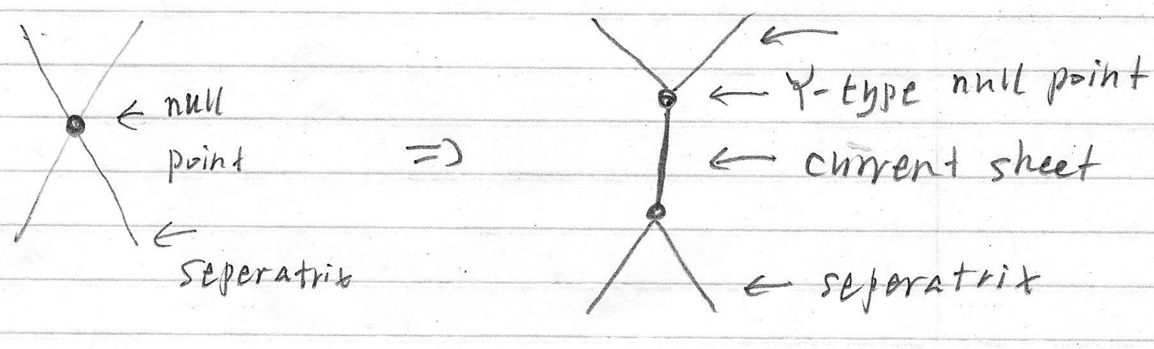


Fig. 2.2

Current sheet solution in 2-D potential field

CH 2-2 (CH2.2.1, CH2.2.2, CH2.2.3)

— use complex variable technique

(x, y) plane ⇒ $z = x + iy$ complex plane

— current sheet is a cut in the complex plane

2-D potential field

$$\nabla \cdot \vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \quad \text{--- (2.9)}$$

$$\nabla \times \vec{B} = 0 \Rightarrow \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 \quad \text{--- (2.10)}$$

Introduce:

$B_y + i B_x = f(z)$

* If $f(z)$ is an analytic function of z,
or $f'(z) = \frac{df}{dz}$ is finite, (B_x, B_y) is potential

** prove:

$$\frac{\partial}{\partial x}: \quad \frac{\partial f(z)}{\partial x} = \frac{df(z)}{dz} \cdot \underbrace{\frac{\partial z}{\partial x}}_{=1} = f'(z)$$

$$\frac{\partial f(z)}{\partial x} = \frac{\partial B_y}{\partial x} + i \frac{\partial B_x}{\partial x} = f'(z) \quad \text{---- (A)}$$

$$\frac{\partial}{\partial y}: \quad \frac{\partial f(z)}{\partial y} = \frac{df(z)}{dy} \cdot \underbrace{\frac{\partial z}{\partial y}}_{i} = i f'(z)$$

$$\frac{\partial f(z)}{\partial y} = \frac{\partial B_y}{\partial y} + i \frac{\partial B_x}{\partial y} = i f'(z) \quad \text{--- (B)}$$

$$(A) * i = (B) \Rightarrow i \frac{\partial B_y}{\partial x} + i \cdot i \frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y} + i \frac{\partial B_x}{\partial y}$$

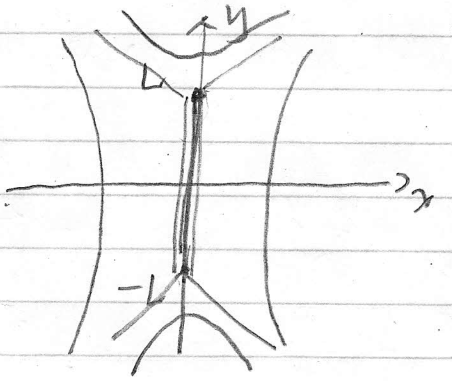
Real part $\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \Rightarrow \nabla \cdot \vec{B} = 0$

Imaginary part $\partial B_y/\partial x - \frac{\partial B_x}{\partial y} = 0 \Rightarrow \nabla \times \vec{B} = 0$

The initial state : X-point

$$B_y + i B_x = z \iff \begin{cases} B_y = x \\ B_x = y \end{cases}$$

The current sheet with two Y-point at $z = -iL$, $z = iL$ on the Y-axis



$$B_y + i B_x = (z^2 + L^2)^{\frac{1}{2}}$$

* $z \gg L$. points far away from current sheet

$$B_y + i B_x = z \implies \text{potential field}$$

* $z = \pm iL$

$$B_y + i B_x = 0 \implies \begin{cases} B_x = 0 \\ B_y = 0 \end{cases} \quad \text{null point}$$

* Along X-axis. $z = x_0$, $x_0 \ll L$

$$B_y + i B_x = \sqrt{L^2 + x_0^2} \implies \begin{cases} B_x = 0 \\ B_y = \sqrt{L^2 + x_0^2} \end{cases}$$

* Along Y-axis. $z = i y_0$. $y_0 \ll L$

$$B_y + i B_x = \sqrt{L^2 - y_0^2} \implies \begin{cases} B_x = 0 \\ B_y = \sqrt{L^2 - y_0^2} \end{cases}$$

large at $y_0 = 0$

zero at $y_0 = \pm L$

* Along Y-axis $z = i(L + y_0)$, $y_0 > 0$, $y_0 \ll L$, the null points

$$z^2 + L^2 = -(L + y_0)^2 + L^2 = \cancel{-L^2} - 2Ly_0 - y_0^2 + \cancel{L^2} = -2Ly_0$$

$$\implies \begin{cases} B_x = \sqrt{2Ly_0} \\ B_y = 0 \end{cases}$$

, the ~~points~~ field beyond null-points // x.

* ~~magnetic~~ electric current in the current sheet

$$\vec{j} = \frac{1}{\mu}\nabla\times\vec{B}$$

$$j_z = \frac{1}{\mu}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \frac{1}{\mu}\frac{\partial B_y}{\partial x}$$ ← gradient along X-direct

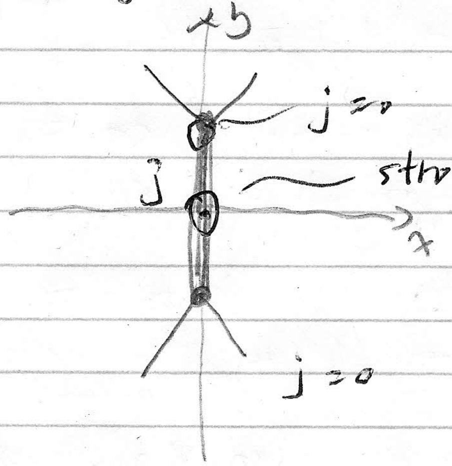
and $\frac{\partial B_x}{\partial y} = 0$, since $B_x = 0$

$$j_z = \frac{1}{\mu}\left[B_y(0^+, y) - B_y(0^-, y)\right]$$

since $B_y = \sqrt{L^2 - y^2}$ along Y-axis

$j_z(y) = \frac{2}{\mu}\sqrt{L^2 - y^2}$: Largest at $y = 0$, the center

0 at $y = L$, the null points



* current sheet with singularity end-points

where it: $$B_y + iB_x = \frac{z^2+a^2}{(z^2+L^2)^{\frac{1}{2}}} \quad \text{----} \quad (2.16)$$

where $a^2 < L^2$

* null points $z = \pm ia$

* singular points, $z = \pm iL$, $(z^2+L^2)^{\frac{1}{2}} \to 0$

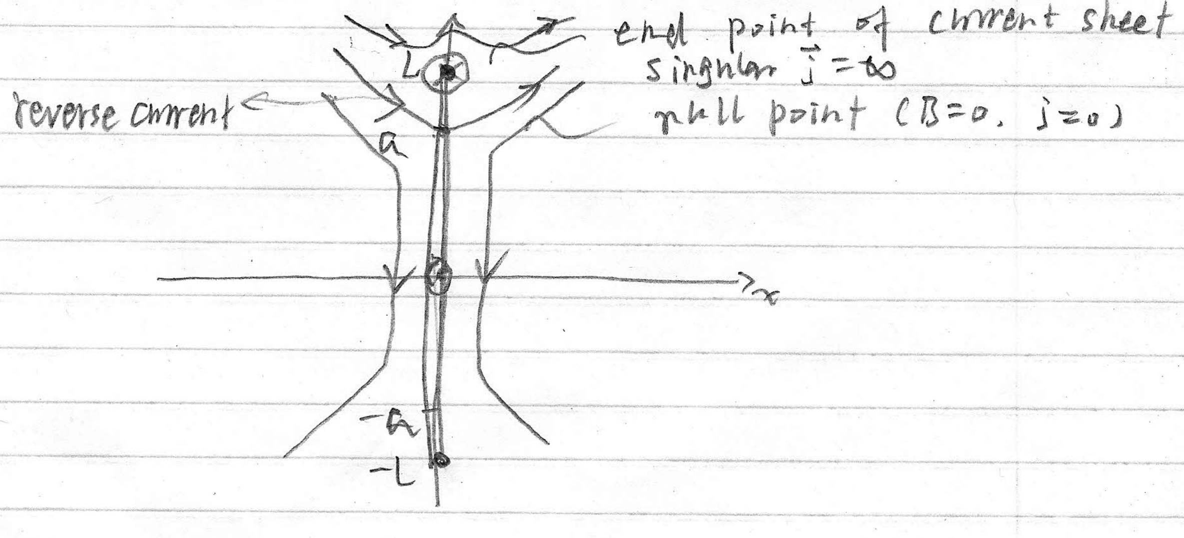
field becomes infinite

* Along the y-axis, $y^2 < L^2$, $z = \pm iy$

$$\frac{z^2+a^2}{(z^2+L^2)^{\frac{1}{2}}} = \frac{a^2-y^2}{(L^2-y^2)^{\frac{1}{2}}} \Rightarrow \begin{cases} B_x = 0 \\ B_y = \dfrac{a^2-y^2}{(L^2-y^2)^{\frac{1}{2}}} \end{cases}$$

* $$j = \frac{1}{\mu}\left[B_y(0^+,y) - B_y(0^-,y)\right] = \frac{1}{\mu}\frac{dB_y}{dx}$$

$$j = \frac{2(a^2-y^2)}{\mu(L^2-y^2)^{\frac{1}{2}}} \quad \text{------}$$

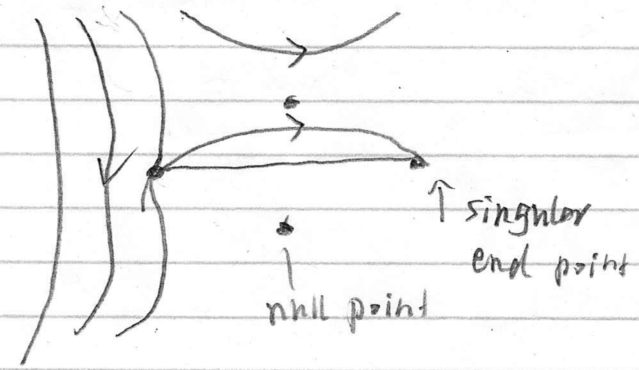


generalized solution

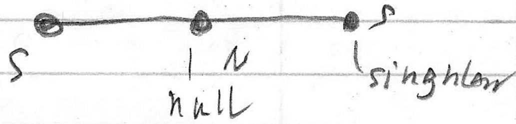
$$B_y + iB_x = -B_0\left[\frac{(b+\frac{1}{2})d^2 + 2dcz - z^2}{\sqrt{z^2-d^2}}\right]$$

* singular end points $d = \pm 1$, current sheet in between
* symmetric null points $c = 0$
* b: position of null points

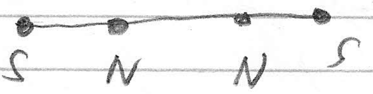
$c=0,\ d=1,\ b=-1.0$



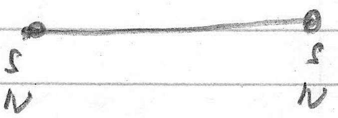
$c=0,\ d=1,\ b=-0.5$



$c=0,\ d=1\ \ b=0$



$c=0,\ d=1\ \ b=0.5$



return to Y-point config.

2D potential field + current sheet in corona

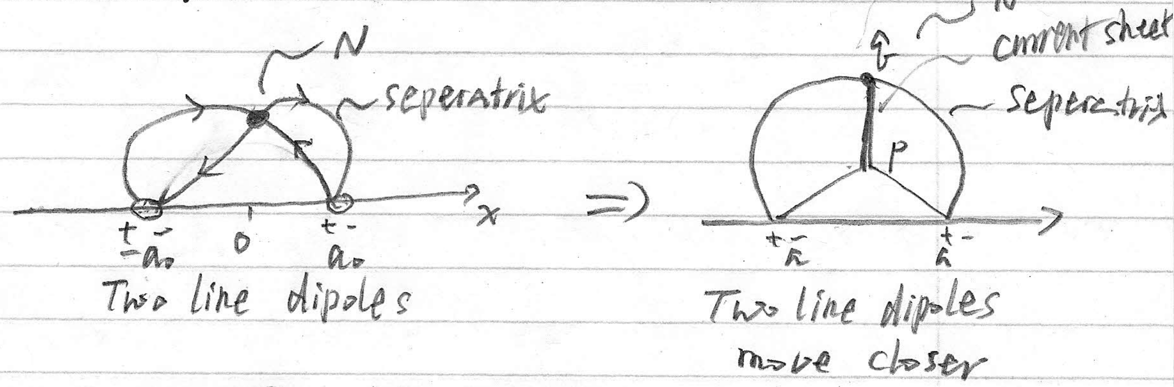


Fig. 2.5

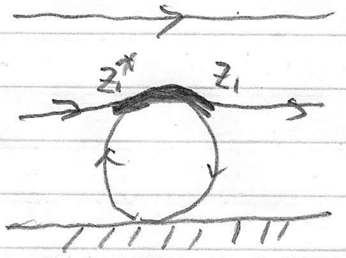
X-point with two line dipoles

$$B_{0y} + iB_{0x} = \frac{iD}{(z+a_0)^2} + \frac{iD}{(z-a_0)^2}$$

Current sheet formation after two dipoles move closer

$$B_y + iB_x = \frac{\bar{D}(z^2+p^2)^{\frac{1}{2}}(z^2+q^2)^{\frac{1}{2}}}{(z^2-a^2)^2}$$

Flux emergence into an existing field



$$B_x - iB_y = \frac{\bar{K}\left[(z^2-z_1^2)(z^2-z_1^{*2})\right]^{\frac{1}{2}}}{z^2}$$

Fig. 2.7