Sep. 30, 2010 ①

Force - Free Field

Aschwanden CH 5.3

The force in $\vec{B}$ field is "Lorentz Force"

$$\vec{F} = \vec{J} \times \vec{B} \; ; \quad \vec{J} : \text{current density}$$

$$\vec{J} = \frac{q}{c} \cdot n\vec{v} \; ; \quad \vec{v} : \text{velocity of charged particles}$$

Lorentz force of single particle

$$\vec{F_q} = \frac{q}{c} \vec{v} \times \vec{B}$$

In plasma $\vec{F} = n\vec{F_q} = \frac{q}{c} n\vec{v} \times \vec{B} = \vec{j} \times \vec{B}$

In plasma : $\vec{J} \times \vec{B}$ is also called "self - force"
or "Lorentz self-force"

In hydrostatic corona ($\vec{V} = 0$), Lorentz force is ~ zero

$$\vec{J} \times \vec{B} = 0$$

since $\rho \frac{d\vec{v}}{dt} = -\nabla p + \vec{J} \times \vec{B}$

and thermal pressure gradient force is negligible
compared with Lorentz force

$$\nabla p \sim 0 \; , \quad \nabla p \ll \vec{j} \times \vec{B}$$

$$\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} = 0$$

Therefore : $\boxed{\text{Force - Free - Field}}$

$$\vec{J} = \frac{1}{4\pi} \nabla \times \vec{B}$$

$$\Rightarrow (\nabla \times \vec{B}) \times \vec{B} = 0 \quad \text{--- --- (5-3.5)}$$

We need to solve this non-linear equation.
General analytic solution does not exist

~~Linear~~ Force Free Field – α parameter

~~The equation can be turned into "Linear"~~

$$\nabla \times \vec{B} = \cancel{4\pi \vec{J}} \; \alpha(\vec{r})\, \vec{B}$$

in order to satisfy $(\nabla \times \vec{B}) \times \vec{B} = 0$

$\alpha(\vec{r})$ is a scalar function of position $\vec{r}$

α: called $\boxed{\text{Force-Free Parameter}}$, is a measure of non-potentiality of $\vec{B}$ field.

$\alpha = 0$, potential field

$\alpha \neq 0$, non-potential

$$\alpha = \frac{\nabla \times \vec{B}}{\vec{B}}, \quad \alpha = \frac{4\pi \vec{J}}{\vec{B}} = \frac{4\pi |\vec{J}|}{|\vec{B}|}$$

– α is not completely arbitrary; α is constant along a magnetic field line

$\nabla \cdot (\nabla \times \vec{B}) = 0$ ⇐ vector identity

$\Rightarrow \nabla \cdot (\alpha \vec{B}) = 0$

$\Rightarrow \alpha \cancel{(\nabla \cdot \vec{B})} + \vec{B} \cdot \nabla \alpha = 0$

$\Rightarrow \vec{B} \cdot \nabla \alpha = 0$

$\Rightarrow$ There is no change of α along $\vec{B}$

Linear Force-Free Field: α constant everywhere

The $\vec{B}$ equation turns into a linear equation

$$\nabla \alpha = 0$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times (\alpha \vec{B}) = \underbrace{\nabla \alpha}_{=0} \times \vec{B} + \alpha (\nabla \times \vec{B})$$

$$\Rightarrow \nabla \times (\nabla \times \vec{B}) = \alpha (\nabla \times \vec{B}) = \alpha^2 \vec{B} \quad \text{--- (5-3.9)}$$

On the other hand: the vector identity

$$\nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} \quad \text{--- (5-3.10)}$$

$$\Rightarrow \nabla \times (\nabla \times \vec{B}) = -\nabla^2 \vec{B}$$

Therefore,

$$\alpha^2 \vec{B} = -\nabla^2 \vec{B},$$

$$\boxed{\nabla^2 \vec{B} + \alpha^2 \vec{B} = 0} \quad \text{——— (5-3.11)}$$

Helmholtz equation

which can be solved numerically from boundary condition

(1) Fourier series

(2) Green's function

(3) spherical harmonics

Non-linear Force-Free Field

α is not constant everywhere : $\nabla\alpha \neq 0$

~~magnetic equation~~

$$\nabla\times(\nabla\times\vec{B}) = \nabla\times(\alpha\vec{B}) = \alpha\underbrace{(\nabla\times\vec{B})}_{\alpha\vec{B}} + \nabla\alpha\times\vec{B}$$

$$\nabla\times(\nabla\times\vec{B}) = \alpha^{2}\vec{B} + \nabla\alpha\times\vec{B}$$

The control equations

$$\boxed{\begin{aligned} \nabla^2\vec{B} + \alpha^2\vec{B} &= \vec{B}\times\nabla\alpha(r) \\ \vec{B}\cdot\nabla\alpha(r) &= 0 \end{aligned}}$$

Need to find both $\vec{B}$ and α

Numerical solutions :

can't be done through global numerical calculation.
e.g. Fourier, harmonic expansion

has to be done through iteration, which has all sorts of numerical problem

1) vertical integration method
Ill-posed problem ; small variation at the boundary can charge the solution dramatically

2) Euler potential method

$$\vec{B} = \nabla\alpha\times\nabla\beta$$

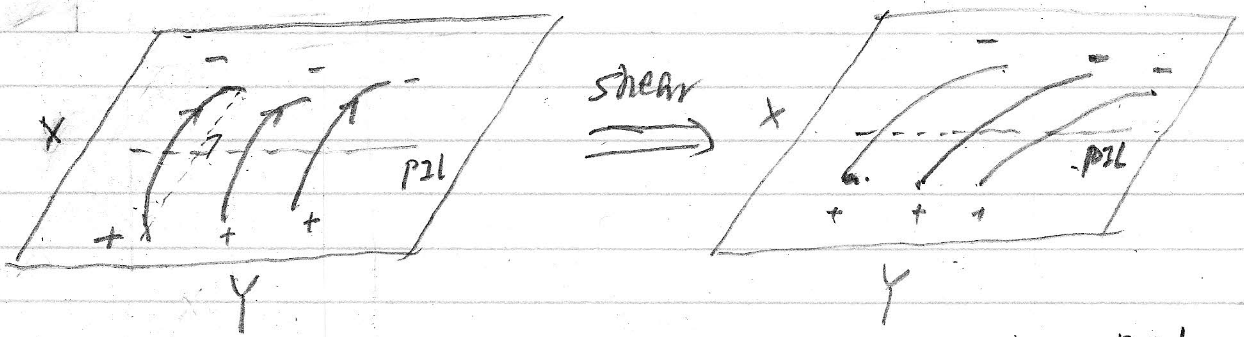
3) Full MHD method

4) Evolution method

$$L = \int_V [B^{-2}|(\nabla\times B)\times B|^2 + (\nabla\cdot\vec{B}|^2)dv$$

minimize $L \Rightarrow (\nabla\times\vec{B})\times\vec{B} = 0$ and $\nabla\cdot\vec{B} = 0$

Sheared Arcade
— An Analytic Solution of Linear Force-Free Field



shear is caused by footpoint shear motion along PIL

Assuming ① $\vec{B}$ is constant or symmetric along Y-direction
② $\vec{B}$ is periodic along X-direction : $\sin kx$
③ $\vec{B}$ exponentially decay along the Z-direction
④ Total $\vec{B}$ or $|\vec{B}|$ is the same at the Z-plane

$$\begin{cases} B_x = B_{x0} \sin kx \exp(-lz) \\ B_y = B_{y0} \sin kx \exp(-lz) \\ B_z = B_0 \cos kx \exp(-lz) \end{cases}$$

$B_0 = \sqrt{B_{x0}^2 + B_{y0}^2}$ the B_0 at $z=0$

k, l : the constants that quantify the fields

Using $\nabla \times \vec{B} = \alpha \vec{B}$

$$\nabla \times \vec{B} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(\nabla \times \vec{B})_x = \cancel{\frac{\partial B_z}{\partial y}} - \frac{\partial B_y}{\partial z} = l B_{y0} \sin kx \exp(-lz)$$

$$(\nabla \times \vec{B})_y = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = (-l B_{x0} + k B_0) \sin kx \exp(-lz)$$

$$(\nabla \times \vec{B})_z = \frac{\partial B_y}{\partial x} - \cancel{\frac{\partial B_x}{\partial y}} = k B_{y0} \cos kx \exp(-lz)$$

$$\Rightarrow \begin{cases} l B_{y_0} = \alpha B_{x_0} \\ -l B_{x_0} + k B_0 = \alpha B_{y_0} \\ k B_{y_0} = \alpha B_0 \end{cases}$$

$$\Rightarrow \begin{cases} B_{x_0} = \frac{l}{k} B_0 \\ B_{y_0} = \frac{\alpha}{k} B_0 \\ k^2 - l^2 - \alpha^2 = 0 \end{cases}$$

If k is given, $k = \frac{1}{L}$. L: the dimension of AR

α is given

one abtain l $\Rightarrow$ how fast field decay along z

$l^2 = k^2 - \alpha^2$

$\alpha \uparrow$, $l \downarrow$, field decrease slower along z

$\alpha \downarrow$ $(\alpha = 0)$, $l \uparrow$, field decrease faster along z

$\alpha \downarrow \rightarrow B \downarrow$ at z, $\rightarrow$ smaller magnetic energy

Shear angle

$$\tan\theta = \frac{B_y}{B_x} = \frac{B_{y_0}}{B_{x_0}} = \frac{\alpha}{l}$$

$\alpha \uparrow$, $\theta \uparrow$,

or vise versa

$\alpha \downarrow$, $\theta \downarrow$

Large shear angle, means large force-free parameter α,

$\Rightarrow$ Large non-potentiality.

θ: can be observed from coronal images

α: can be calculated from vector-magnetogram obs.

consider z component of $(\nabla \times B)_z = \alpha B_z$

$$\alpha(x, y, z=0) = \frac{1}{B_z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$\Rightarrow$

Obtain the field line function

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{B_{y0}}{B_{x0}} = \frac{\alpha}{l} = \tan\theta$$

$$\frac{dz}{dx} = \frac{B_z}{B_x} = \frac{B_0 \cos kx}{B_{x0} \sin kx} = \frac{k}{l}\frac{\cos kx}{\sin kx}$$

Using vector property:

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}$$

$$\begin{cases} y(x) = \frac{\alpha}{l} x + y_0 & \text{--- (5-3.19)} \\ z(x) = \frac{k}{l} \log[\sin(kx)] + z_0 \end{cases}$$

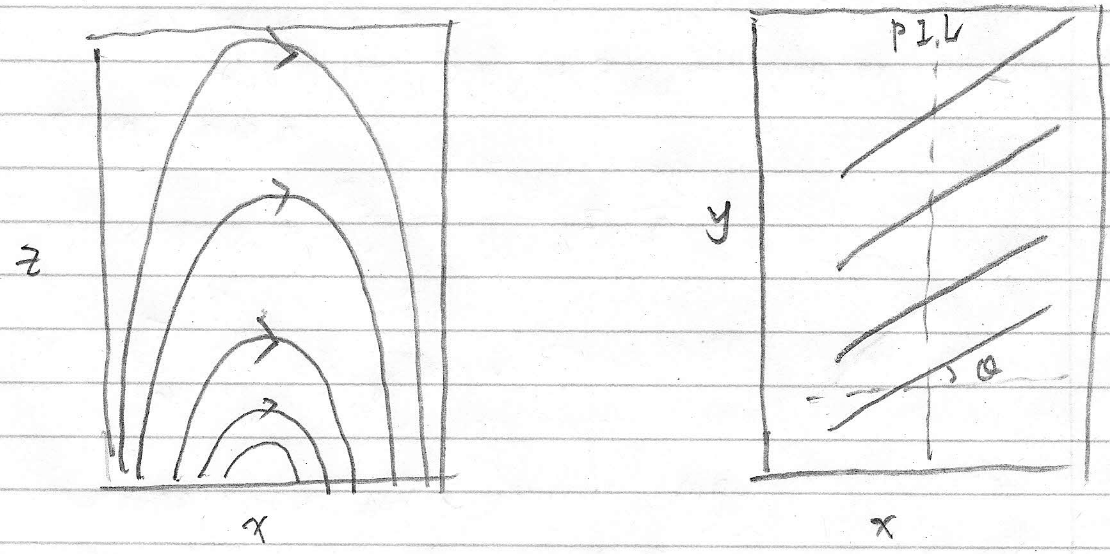
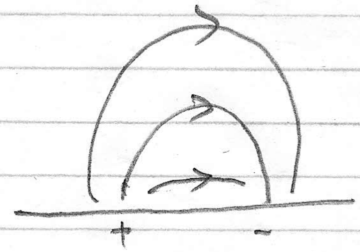


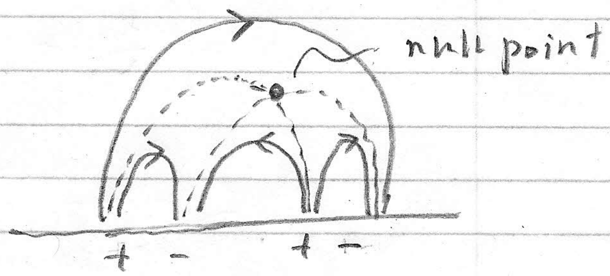
Figure 5-4

Null points and Separators.

— study of magnetic topology if mixed polarities



dipole:
no null points



quadraple.
• null point
---- : seperatrix curve
-—- : seperatrix surface in 3D

* Magnetic null points are single-point locations where magnetic field vanishes : $\vec{B} = (0,0,0)$
  - the weakest link.
  - where magnetic reconnection suppose to occur

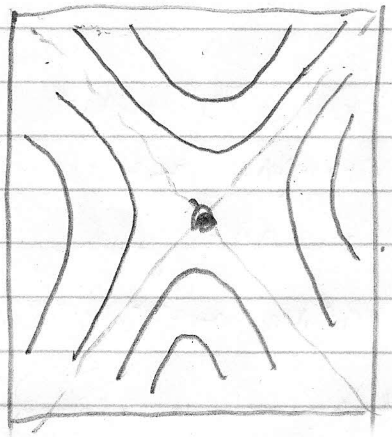
* separatrix : seperate different topological domain
  - surface in 3D
  - curve in 2D

* topological domain : a family of magnetic field lines that connect to the same conjugate pair of magnetic polarities.
  - Dipole : one domain, no null point, ~~one separatrix surface at infinite~~
  - Two Dipole (quadruple)

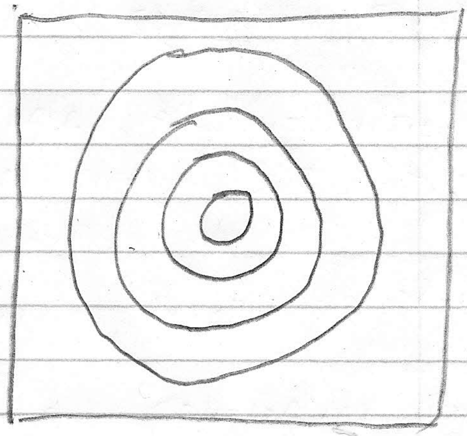
four domains, four separatrix surfaces

$\Rightarrow$

* Separator lines: the intersection of two separatrix surface
* separator lines connect two null points
* Two types of null points:
  1) X-point
  2) O-point (at the center of magnetic island)



X-point



O-point

Figure 5.26