Sep. 16, 2010

Potential magnetic field

Aschwanden CH 5.2

— the simplist form of magnetic field

$$\vec{B} = \nabla\phi \quad \text{--- (5-2.2)}$$

ϕ: scalar magnetic potential field

"−" or "+"

In notation of Aschwanden, in cgs unit

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = 4\pi \vec{j} \qquad \nabla \times \vec{B} = \mu_0 \vec{j}$$

cgs (Aschwanden) — mks (Priest & Forbes)

$$\vec{j} = \frac{1}{4\pi}(\nabla \times \vec{B}) = \frac{1}{4\pi}(\nabla \times \nabla\phi) = 0$$

Because $\nabla \times \nabla\phi = 0$ for any ϕ

Therefore $\boxed{\vec{J} = 0}$ in potential field (5-2.6)

also called $\boxed{\text{current-free field}}$

Since $\nabla \cdot \vec{B} = 0$

$$\nabla \cdot (\nabla\phi) = 0$$

$$\boxed{\nabla^2 \phi = 0} \quad \text{potential field equation}$$

$$\rightarrow \phi \rightarrow \vec{B}$$

In Cartesian: $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = \nabla^2\phi = 0$

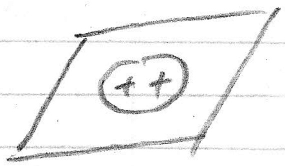
In spherical: $\nabla^2\phi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\phi}{\partial\theta}\right)$

$$+ \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2}$$

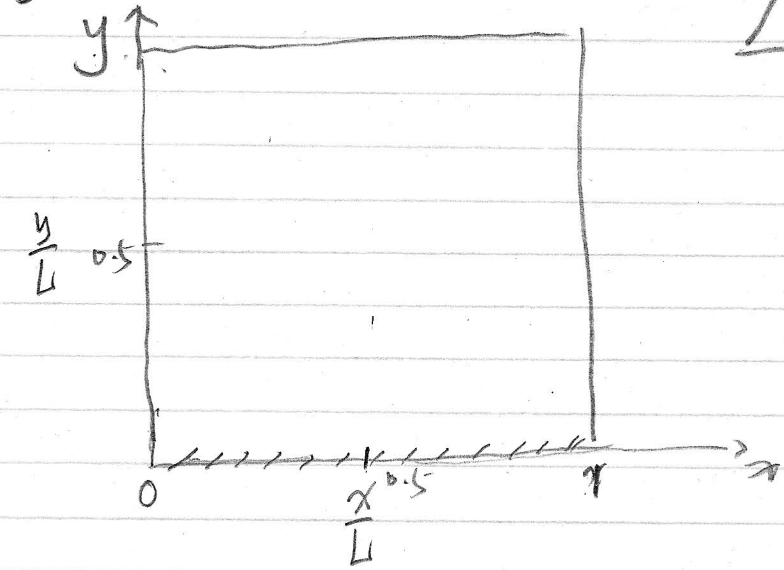
Unipolar field solution (Aschwanden CH 5-2.1)

- 2D solution
- model the coronal magnetic field of a sunspot (symmetric)

$\nabla^2 \phi = 0$



boundary condition : L × L box



$$\begin{cases} \phi(x,0) = \text{symmetric to } (\frac{x}{L} = 0.5, \; x = \frac{L}{2}) & \\ \phi(x,\infty) = 0 & \text{okay} \\ \phi(0,y) = 0 & \text{strong assumption} \\ \phi(L,y) = 0 & \text{strong assumption} \end{cases}$$

$$\nabla^2 \phi = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Using separation of dimension to find analytic solution

$\phi(x,y) = X(x) Y(y)$ — assumption

Consider $X(x)$ = potential along x-direction

$X(x=0) = X(x=L) = 0$ and symmetric

One instance of solution

$X(x) = X_0 \sin(\pi \frac{x}{L})$: 0 at $x=0$, $x=L$; max at $x = \frac{L}{2}$, the middle

(3)

Considering $Y(y)$, one instance of solution is

$$Y(y) = Y_0 \exp\left(-\frac{y}{y_0}\right)$$

$y=0$ highest

$y \to \infty$ $Y(y=\infty) = 0$

$$\nabla^2 \phi = \frac{\partial^2 X}{\partial x^2} Y + X \frac{\partial^2 Y}{\partial y^2}$$

$$= \left(-\frac{\pi^2}{L^2} + \frac{1}{y_0^2}\right) XY = 0$$

Therefore, $-\frac{\pi^2}{L^2} + \frac{1}{y_0^2} = 0$

$$\Rightarrow y_0 = \frac{L}{\pi}$$

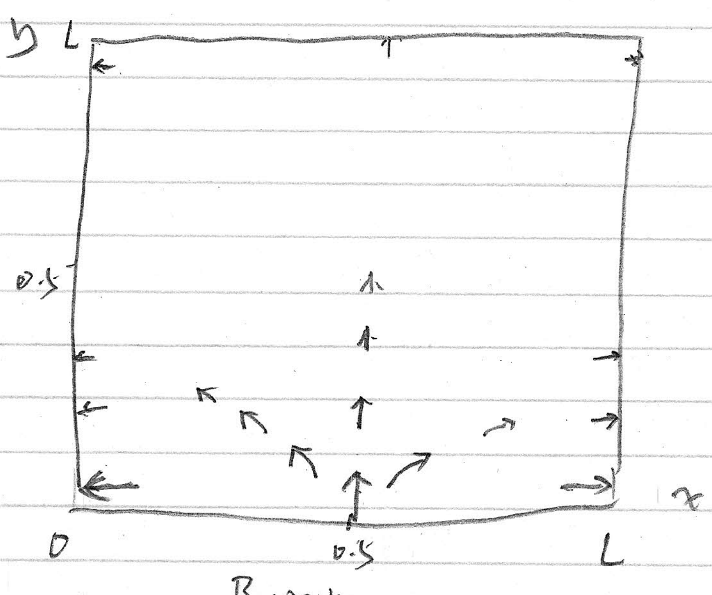
One solution is

$$\phi(x,y) = \phi_0 \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\pi y}{L}\right)$$

$$B_x(x,y) = +\frac{\partial \phi}{\partial x} = +B_0 \cos\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\pi y}{L}\right)$$

$$B_y(x,y) = +\frac{\partial \phi}{\partial y} = -B_0 \sin\left(\frac{\pi x}{L}\right) \exp\left(-\frac{\pi y}{L}\right)$$

$$B(x,y) = \sqrt{B_x^2 + B_y^2} = B_0 \exp\left(-\frac{\pi y}{L}\right)$$



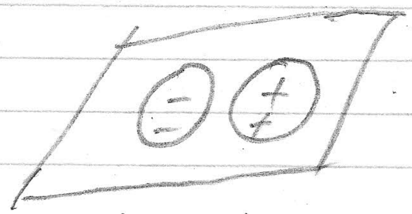
$B(x,y)$

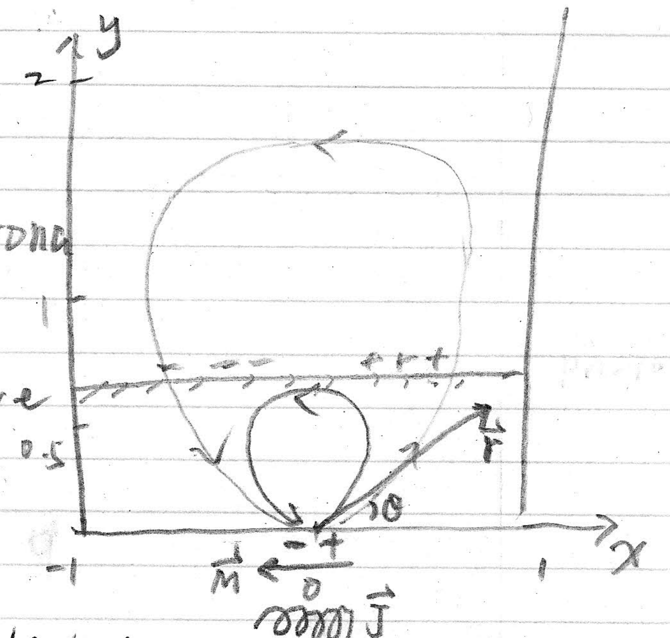
At the lower boundary, $y=0$

$B(x,0) = B_0$;

the same magnitude

* Approximate the dipole field in 2-D
— Aschwanden Book CH 5-2-2





$y=0$: buried dipole $\vec{M}$
at $x=0$, $y=0$

Again, using magnetic potential
* For point charge (if exist) or monopole

$$\phi = \frac{q}{r} \quad \text{from Gauss's Law}$$

$$B = \nabla\phi = -\frac{q}{r^2}\hat{r}. \quad \hat{r}: \text{unit vector of } \vec{r}$$

in polar coordinate

* For ~~most~~ dipole, or far field approximation of (− +)

$$\phi = \frac{\vec{M}\cdot\vec{r}}{r^3} = \frac{-m r\cos\theta}{r^3} = -\frac{m\cos\theta}{r^2} \quad \text{---(5-2.17)}$$

$\vec{m}$: in the direction of $-\hat{x}$, or $\theta = \pi$
Corresponding to a current coil buried at $x=0$, $y=0$
Equivalent to $\boxed{m = \frac{\pi a^2 I}{c}}$

$$\vec{B}(\vec{r}) = (B_r, B_\theta) = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \quad \text{in 3D. } B_\varphi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$$

$$B_r = \frac{2m\cos\theta}{r^3}, \quad B_\theta = \frac{m\sin\theta}{r^3} \quad \text{---- (5.2.15)}$$

From ~~Bolar~~ Polar to Cartisian Coordinates

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B_r \\ B_\theta \end{pmatrix}$$

$$B_x = B_r \cos\theta - B_\theta \sin\theta$$
$$B_y = B_r \sin\theta - B_\theta \cos\theta$$

\# One could draw arrows to indicate ~~arrows~~, vectors
one could also draw streamlines of vectors

Find equation of magnetic field lines
using the proportionality relation of a vector

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{ds}{B} \qquad \text{--- (5.2.19)}$$

s : distance along the direction of field line

In spherical or polar coordinates

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{ds}{B}$$

$$\frac{dr}{d\theta} = r\frac{B_r}{B_\theta} = r \cdot \frac{2m\cos\theta / r^3}{m\sin\theta / r^3} = \frac{2r\cos\theta}{\sin\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\,d\theta = \frac{2}{\sin\theta}\,d\sin\theta$$

$$\ln r \Big|_{r_1}^{r} = 2\ln\sin\theta \Big|_{\sin\theta=1}^{\sin\theta}$$

$$\frac{r}{r_1} = \sin^2\theta$$

$$r = r_1 \sin^2\theta \qquad \text{--- (5.2.22)}$$

Different r_1 . different field lines

Method of potential field calculation.

— using observed line of sight (LOS) magnetic field as the boundary condition. B_n

B_n could be anything: monopole, dipole, or mixed polarities

Also, consider the global magnetic field

Control Functions: $\nabla^2 \phi = 0 \quad (z > 0)$ ——— (5-2-23)

z: solar surface

Boundary condition: Neumann boundary (versus Dirichlet boundary $\phi = \phi(r > 0)$)

Lower boundary: $-\hat{n} \cdot \nabla \phi = B_n \quad (z = 0)$ (5-2-24)

B_n: from observation B_{LOS}

$B_n = B_{LOS}$ at the disk center

$B_n \neq B_{LOS}$ close to limb, due to projection effect

B_n: normal component, or longitudinal component

Upper boundary $\lim_{r \to \infty} \phi(r) = 0$ ——— (5-2-25)

$\boxed{\text{These equations have to be solved numerically}}$

Method 1: Green's Function Method

Method 2: Eigen function Expansion Method

Green Function Method

A point "magnetic charge" produce a potential of

$G_n(\vec{r}, \vec{r}') = \frac{1}{2\pi |\vec{r} - \vec{r}'|}$, $\vec{r}'$: location of point charge

$\phi(r) = \iint B_n(\vec{r}')\, G_n(\vec{r}, \vec{r}')\, dx' dy'$ - - - (5-2-27)

This function satisfies the $\nabla^2 \phi = 0$ and the boundary condition

Eigenfunction Expansion Method

useful for global whole sun calculation

(Altschuler & Newkirk, 1969)

Now, 3D-spherical coordinates are the natural one

(r, θ, φ)

Control function

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

Lower boundary:

$$-\hat{r} \cdot \nabla \phi = B_L(\theta, \varphi) \quad \text{at } r = 1 R_\odot$$

Upper boundary

$$\phi = 0 \quad \text{at} \quad r = r_w$$

r_w: the so-called source surface, typically $r_w = 2.5 R_\odot$

At r_w ① $\phi = 0$

② All magnetic field points in the radial direction

③ The true surface of interplanetary $\vec{B}$

true source surface

Thus $\boxed{\text{PFSS}}$ model

$\boxed{\text{Potential field Source Surface model}}$

General solution of Laplacian $\nabla^2 \phi = 0$ in spherical coords.

$$\phi(\vec{r}, \theta, \varphi) = R_\odot \sum_{l=1}^{N} \sum_{m=0}^{l} f_l(r)\, P_l^m(\theta) \left(g_l^m \cos m\varphi + h_l^m \sin m\varphi\right)$$

— (5.2.35)

The separation of dimensions in each (l, m) mode

Along r:

$$f_l(r) = \frac{(r_w/r)^{l+1} - (r/r_w)^l}{(r_w/R_\odot)^{l+1} - (R_\odot/r_w)^l}$$

At $r = r_w$, $f_l(r) = 0$

At $r = r_w$, the source surface

$f_\ell(r = R_0) = 1$; $f(r)$ is normalized

$P_l^m(\theta)$, the Legendre polynomials

	$m=0$	$m=1$	$m=2$
$L=0$	$P_L^m(\cos\theta) = 1$	NA	NA
$L=1$	$P_1^0 = \cos\theta$	$P_1^1 = \sin\theta$	NA
$L=2$	$P_2^0 = \frac{1}{2}(3\sin^2\theta - 1)$	$P_2^1 = -3\sin\theta\cos\theta$	$P_2^2 = 3\sin^2\theta$

$$P_L^0(x) = \frac{1}{2\pi i}\oint (1 - 2tx + t^2)^{-\frac{1}{2}} t^{-L-1}\, dt$$

$$P_L^m(x) = (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_L(x)$$

The unknowns are

① g_l^m

② h_l^m

③ To which order ℓ the calculation

which can be found from the boundary condition

$$B_n = (\nabla\phi)_n = \frac{\partial}{\partial r}\phi(r, \theta, \varphi) \quad \text{at } r = R_0 \text{ --- (A)}$$

$$B_n(\theta, \varphi) = R_0 \sum_{\ell=1}^{N} \sum_{m=0}^{\ell} K_\ell(R_0) P_l^m(\theta)\left(g_l^m \cos m\varphi + h_l^m \sin m\varphi\right) \text{ --- (A)}$$

$$K(r) = \left.\frac{d f_\ell(r)}{dr}\right|_{r=r_0} = \frac{-\frac{L+1}{r_w}\left(\frac{r_w}{R_0}\right)^{L+2} - \frac{L}{r_w}\left(\frac{R_0}{r_w}\right)^{L-1}}{\left(\frac{r_w}{R_0}\right)^{L+1} - \left(\frac{R_0}{R_w}\right)^{L}}$$

To find $g_{l'}^{m'}$, multiply (A) Eq by $P_{l'}^{m'}(\theta)\cos m'\varphi$ and integrate over $\int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi$

For all m' for each ℓ

For all ℓ' for 0 to N. ~~(N = 400)~~

For order l', m', the Left Hand Side (LHS)

$$LHS = \frac{1}{4\pi}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} B_n(\theta,\varphi)\, P_{l'}^{m'} \cos m'\varphi \cdot \sin\theta \, d\theta \, d\varphi$$

$$RHS = R_\odot \frac{1}{4\pi}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \sum_{l=1}^{N}\sum_{m=0}^{l} K_l(R_\odot) P_l^m(\theta)\left(g_l^m \cos m\varphi + h_l^m \sin m\varphi\right) \cdot P_{l'}^{m'}(\theta) \cos m'\varphi \, \sin\theta \, d\theta \, d\varphi$$

Since $\frac{1}{4\pi}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} P_l^m(\theta)\, P_{l'}^{m'}(\theta) \cos m\varphi \cos m'\varphi \, \sin\theta \, d\theta \, d\varphi$

$$= \frac{1}{2l+1}\delta_{ll'}\delta_{mm'} \qquad \text{Legendre Polynomials}$$

$$RHS = \frac{R_\odot}{2l'+1} K_{l'}(R_\odot)\, g_{l'}^{m'}$$

$$\therefore \; g_{l'}^{m'} = \frac{2l'+1}{4\pi} \frac{1}{R_\odot K_{l'}(R_\odot)} \int_{\theta=0}\int_{\varphi=0} B_n(\theta,\varphi) P_{l'}^{m'}(\theta) \cos m'\varphi \cdot \sin\theta \, d\theta \, d\varphi$$

OR

$$g_l^m = \frac{2l+1}{4\pi} \frac{1}{R_\odot K_l(R_\odot)} \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} B_n(\theta,\varphi)\, P_l^m(\theta) \sin\theta \cos m\varphi \, d\theta \, d\varphi$$

Magnetogram resolution: 5°, or 72×36 pixels, global neutral line

1°, or 360×180 pixels, global structure

0.1°, 3600×1800 pixels, fine global structure

Truncation of N

$5^\circ \longleftrightarrow N \sim 30$

$1^\circ \longleftrightarrow N \sim 100$

$0.1^\circ \longleftrightarrow N \sim 300$

Find g_l^m, h_l^m in a numerical way

method 1: Inversion of matrix of $N \times N$

Zhao, X-P. method from Stanford

Method 2: FFT: fast-fourier-transform method from LMSAL by ~~Welsch, R~~ Dr. DeRosa

PFSS package in SSW (Solar ~~soft wa~~ software)

- using method 2, plus 3-D visualization

$SSW/packages/pfss

① create an account in space weather lab computer

"helio.gmu.edu" — first entry point

"magnet.mesa.gmu.edu"

"wind.mesa.gmu.edu"

"corona.mesa.gmu.edu"

"swl.mesa.gmu.edu"