Sep. 02, 2016

MHD Equations — Magneto - Hydro - Dynamic Eq.

Including

① hydrodynamic / Fluid Equations (×3)

①-medium relation : Equation of state (×1)

② Maxwell Equations (×4)

②-medium relation : Generalized Ohm's Laws (×1)

Sep. 9, 2010

Hydrodynamic Equations

① Mass Conservation

$$\frac{d\rho}{dt} + \rho(\nabla\cdot\vec{V}) = 0 \quad \text{-------} \quad (1)$$

or $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$$

ρ = density
$\vec{V}$: velocity

or $$\frac{\partial\rho}{\partial t} + (\vec{V}\cdot\nabla)\rho + \rho(\nabla\cdot\vec{V}) = 0$$

$\frac{d}{dt}$, i.e. $\frac{d\rho}{dt}$, "convective derivative": derivative for a parcel moving together with flow $\vec{V}$.
Also called "total derivative"

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\vec{V}\cdot\nabla)\rho \, ,$$ ρ can be any variable

where $\frac{\partial\rho}{\partial t}$: "local derivative" or "partial derivative"
derivative of a parcel fixed in space

$(\vec{V}\cdot\nabla)\rho$: gradient of the variable along the flow field

$\nabla\cdot\vec{V} = -\frac{1}{\rho}\frac{d\rho}{dt}$: the rate of density change, ~~(normalized)~~ normalized with the density

② Momentum ~~eq~~ Conservation

$$\rho\frac{d\vec{V}}{dt} = -\nabla P + \vec{J}\times\vec{B} + \nabla\cdot\vec{\vec{S}} + \vec{F_g} \quad \text{---} \quad (2)$$

$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

where, P : plasma thermal pressure ; surface force
$\vec{J}\times\vec{B}$: Lorentz force ; volume force

$\vec{J}$: electric current density

$\vec{B}$: magnetic induction intensity.

$$\vec{B} = \mu_0 \vec{H}$$

μ_0 : magnetic permeability in free space ($4\pi \times 10^{-7}\, H M^{-1}$)

$\vec{H}$: magnetic field intensity

$\vec{\vec{S}}$: fluid viscous stress tensor

$\nabla \cdot \vec{\vec{S}}$: fluid viscous force caused by velocity gradient / shear

$$\nabla \cdot \vec{\vec{S}} = \rho \nu \nabla^2 \vec{V} + \rho (\xi + \tfrac{2\nu}{3}) \nabla (\nabla \cdot \vec{V})$$

ν : coefficient of kinematic shear

ξ : coefficient of bulk viscocity

In space plasma, viscosity is often negligible

$$\nabla \cdot \vec{\vec{S}} = 0$$

$\vec{F_g}$: gravitational force

Energy Conservation

$$\rho \frac{de}{dt} + P \nabla \cdot \vec{V} = \nabla \cdot (\vec{\vec{K}} \cdot \nabla T) + (\vec{\vec{\eta}}_e \cdot \vec{j}) \cdot \vec{j} + Q_v - Q_r$$

e : internal energy

$$e = \frac{P}{(\gamma - 1)\rho}$$

γ : ratio of specific heats $\gamma = \frac{C_p}{C_v}$

$\gamma = \frac{5}{3}$ ideal gas

$1 < \gamma < \frac{5}{3}$: typical space plasma

$P \nabla \cdot \vec{V}$: work done by fluid expansion or contraction

$\nabla \cdot \vec{V} = 0$ Incompressible

* $\nabla \cdot (\overleftrightarrow{K} \cdot \nabla T)$: thermal conduction

  $\overleftrightarrow{K}$: thermal conductivity tensor

  $\overleftrightarrow{K} \rightarrow K$ if isotropic medium ($W\, m^{-1}\, K^{-1}$)

  $\nabla \cdot (\overleftrightarrow{K} \cdot \nabla T) = K \nabla \cdot \nabla T = K \nabla^2 T$

  However, in magnetized plasma $K_\perp \neq K_\parallel$

  $\perp$: perpendicular to $\vec{B}$

  $\parallel$: parallel to $\vec{B}$

* $(\overleftrightarrow{\eta}_e \cdot \vec{J}) \vec{J}$: Joule heating (electric heating

  $\overleftrightarrow{\eta}_e$: electrical resistivity tensor

  $\overleftrightarrow{\eta}_e \rightarrow \eta_e$ if isotropic : resistivity η_e

  (however, $\eta_\parallel \neq \eta_\perp$)

  $$\eta = \frac{1}{\sigma}$$

  σ : conductivity (siemens m^{-1})

  $\eta \vec{J} = \vec{E}$ $\quad \vec{E}$: electric field

  or $\vec{J} = \sigma \vec{E} = \frac{1}{\eta} \vec{E}$ $\quad$ (Ohm's Law)

  $$(\overleftrightarrow{\eta}_e \cdot \vec{J}) \cdot \vec{J} = \vec{E} \cdot \vec{J} = \sigma \vec{E} \cdot \vec{E} = \sigma E^2 = \eta J^2$$

* Q_ν : heating due to viscosity

  $Q_\nu = \vec{V} \cdot (\nabla \cdot \overleftrightarrow{S}) - \nabla \cdot (\overleftrightarrow{S} \cdot \vec{V})$

  In space plasma, $Q_\nu = 0$

* Q_r : radiative loss ; energy loss due to radiation

  $Q_r = \rho^2 Q(T)$

  $Q(T)$: radiative loss function

  i.e. Chianti database

* Equation of state — constitutive relation of the medium

$$P = R\rho T = n k_B T$$

R : universal gas constant $= 8300\ m^2 s^{-2} deg^{-1}$

ρ : mass density (g/cm^3)

n : particle number density $(\#/cm^3)$

k_B : Boltzmann constant ; $= 1.38 \times 10^{-23}\ J/K$
$= 1.38 \times 10^{-16}\ ergs/K$

$$\rho = n \bar{m}$$

$\bar{m}$: mean particle mass

In 100% hydrogen plasma, fully ionized

$$n \simeq n_e + n_p \approx 2n_e = 2n_p$$

$$\cancel{m = m_e + m_p \approx m_p}$$

$$\rho = \rho_e + \rho_p = n_e m_e + n_p m_p = n_p m_p$$

$$\rho = n \bar{m}$$

$$\bar{m} = \frac{\rho}{n} = \frac{n_p m_p}{2 n_p} = \frac{m_p}{2}$$

$$P = \underline{2n_e} k_B T = n k_B T = \frac{\rho}{\bar{m}} k T$$

In polytropic process.

$$P = c\rho^{\Gamma}$$

Γ : polytropic index

For ordinary ideal gas adiabatic process : $\Gamma = \gamma = \frac{5}{3}$

isothermal process : $\Gamma = 1$

For magnetized ideal gas : $\Gamma \neq \frac{5}{3}$ $\quad 1 < \Gamma < \frac{5}{3}$

Maxwell Equations

The general form from experiment in a medium

Gauss's Law: $\nabla \cdot \vec{D} = \rho_c$ ------ (1)

Gauss's Law for magnetism: $\nabla \cdot \vec{B} = 0$ ------ (2)

since $\rho_B = 0$

~~Ampere's Law: $\nabla \times \vec{B} = \vec{J}$.~~

Faraday's Law: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ------ (3) since $\vec{J}_B = 0$

Ampere's Law $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$ ------ (4)

Where D: electric displacement

ρ_c: electric charge density

$\vec{B}$: magnetic induction

$\vec{E}$: electric field

$\vec{H}$: magnetic field

$\vec{J}$: electric current

In a medium ~~of dielectric, isotropic, non-magnetized~~ of isotropic, linear

The medium relations, or constitutive relations

$$\vec{D} = \varepsilon \vec{E} \quad \text{------ (1)}$$

ε: dielectric permitivity

In vacuum, $\varepsilon = \varepsilon_0$ dielectric permitivity in free space

$\varepsilon_0 = 8.854 \times 10^{-12}$ farad m^{-1}

$$\vec{B} = \mu \vec{H}$$

μ: magnetic permeability

$\mu_0 = 4\pi \times 10^{-7}$ Henry m^{-1} in free space

$$\vec{J} = \sigma \vec{E} \qquad (3)$$

Ohm's Law

* In dielectric medium.

No free charge : $\rho_c = 0$

No conductivity : $\sigma = 0$

$J = 0$

The reduced Maxwell's equations in dielectric

$$\nabla \cdot \vec{E} = 0 \qquad (1)$$

$$\nabla \cdot \vec{H} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad (3)$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

which can be used to do $\Rightarrow$

Electro-magnetic wave equations

$$\nabla^2 \vec{E} = \varepsilon \mu_0 \frac{\partial^2 E}{\partial t^2}$$

Snell's Law

Polarization

* what in conducting medium ?

EM class

In plasma,

No net free charge $\rho_c = 0$

$$\varepsilon = \varepsilon_0$$

$$\mu = \mu_0$$

Generalized Ohm's Law

Since $\vec{J} = \sigma \vec{E}'$ in a moving plasma

$\vec{E}'$: electric field in the frame moving with plasma

$$\vec{E}' = \vec{E} + \underbrace{\vec{V} \times \vec{B}}$$

motion-induced electric field

Lorentz transformation

$\vec{E}$: electric field in a laboratory ~~frame of reference~~

$$\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B})$$

Displacement current $\frac{\partial \vec{D}}{\partial t} \ll \vec{J}$, $\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ll \vec{J}$

is negligible in plasma, since $\ll \vec{J}$

Reduced Mx Eqs.

$$\nabla \cdot \vec{E} = 0 \qquad (1)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad (4)$$

Combining (3) and (4), and generalized Ohm's Law to obtain

magnetic induction equation

$$(3) \Rightarrow \frac{\partial \vec{B}}{\partial t} = -(\nabla \times \vec{E})$$

$$\vec{E} = \frac{\vec{J}}{\sigma} - \vec{V} \times \vec{B}$$

$$\nabla \times \vec{E} = \frac{1}{\sigma} \nabla \times \vec{J} - \nabla \times (\vec{v} \times \vec{B})$$

$$\nabla \times \vec{E} = \frac{1}{\sigma \mu_0} [\nabla \times (\nabla \times \vec{B})] - \nabla \times (\vec{v} \times \vec{B})$$

$$\cancel{\nabla \times \nabla \times \vec{B} = (\nabla \cdot \nabla) \vec{B}}$$

From vector identity of

$$\nabla \times (\nabla \times \vec{B}) = \cancel{\nabla (\nabla \cdot \vec{B})} - \nabla^2 \vec{B}$$

$$\nabla^2 \vec{B} = (\nabla \cdot \nabla) \vec{B}$$

Therefore

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\sigma \mu_0} \nabla^2 \vec{B}$$

or

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 B}$$

magnetic induction equation

η: magnetic diffusivity

$$\eta = \frac{1}{\sigma \mu_0} = \frac{\eta_e}{\mu_0}$$

η_e: electric ~~conductivity~~ resistivity

σ: electric conductivity

convection term: $\nabla \times (\vec{v} \times \vec{B})$

This is how magnetic field generated through solar dynamo, or differential motion of the Sun

diffusion term: $\eta \nabla^2 \vec{B}$:

This is how magnetic field dissipated

① through resistivity, but η small

② through reconnection, $\nabla^2 \vec{B}$ large

Magnetic Reynolds number

$$R_m = \frac{\text{convection term}}{\text{Diffusion term}} = \frac{\nabla \times (\vec{v} \times \vec{B})}{\eta \nabla^2 B}$$

Using dimensional analysis and characteristic sclales

$$\nabla = \frac{1}{L_0} \qquad L_0 : \text{characteric length}$$

$$\vec{v} \to V_0 \qquad V_0 : \text{characteristic velocity}$$

$$\nabla \times (\vec{v} \times \vec{B}) = \frac{1}{L_0}(V_0 B_0)$$

$$\nabla^2 B = \frac{1}{L_0^2} B_0$$

$$R_m = \frac{L_0 V_0}{\eta}$$

From Spitzer (1962)

$$\eta = \frac{c^2 e^2 m_e^{\frac{1}{2}}}{3(2\pi)^{3/2} \varepsilon_0} \ln\Lambda \, (K_B T_e)^{-3/2} \quad \text{——} \quad (1.13)$$

$$= 1.05 \times 10^8 \, T_e^{-3/2} \ln\Lambda \quad m^2 s^{-1}$$

$\ln\Lambda$: the Coulomb logarithm

$$\ln\Lambda = \begin{cases} 16.3 + \frac{3}{2}\ln T - \frac{1}{2}\ln n & T < 4.2\times10^5 K \\ 22.8 + \ln T - \frac{1}{2}\ln n & T > 4.2\times10^5 K \end{cases}$$

Exp. Active region corona

$T \sim 10^6 K$, $n = 10^{15} m^{-3}$, $L_0 \sim 10^5 m$, $V_0 = 10^4 m s^{-1}$

$\Rightarrow R_m = 10^9$

$\eta = 1 \, m^2 s^{-1}$.

convection terms dominates. diffusion term negligible

\# Diffusion time scale τ_d

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B}$$

$$\frac{B_0}{\tau_d} = \eta \frac{B_0}{L_0^2}$$

$$\tau_d = \frac{L_0^2}{\eta}$$

For a sunspot, $L_0 = 10^5$ m, $\eta = 1\ m^2 s^{-1}$

magnetic loop, magnetic element

$$\tau_d = 10^{10} \text{ sec} = 300 \text{ years}.$$

Sunspot persists; its dissappearance depends on the surface turbulent motion