Hw #5 — solution

1. 2-D field.

$\vec{B} = (B_x, B_y)$ only, no $\hat{z}$ component, and $\frac{\partial \vec{B}}{\partial z} = 0$

Similarly $\vec{V} = (V_x, V_y)$ only, no $\hat{z}$ component, $\frac{\partial \vec{B}}{\partial z} = 0$

Generalized Ohm's Law

$$\vec{J} = \sigma(\vec{E} + \vec{V}\times\vec{B})$$

or $$\vec{E} = \frac{\vec{J}}{\sigma} - \vec{V}\times\vec{B}$$

From Ampere's Law. $\vec{J} = \frac{1}{\mu}\nabla\times\vec{B}$

$$\vec{E} = \frac{1}{\mu\sigma}\nabla\times\vec{B} - \vec{V}\times\vec{B}$$

(1) prove $\vec{E}$ has only $\hat{z}$ component

check the first term in the RHS (Right Hand Side)

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & \cancel{B_z}_{\,0} \end{vmatrix} = \left(\underbrace{\frac{\partial B_z}{\partial y}}_{0} - \underbrace{\frac{\partial B_y}{\partial z}}_{0}\right)\hat{x} + \left(\underbrace{\frac{\partial B_x}{\partial z}}_{0} - 0\right)\hat{y} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{z}$$

$$\nabla\times\vec{B} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{z}$$ has $\hat{z}$ component only.

check the second term in the RHS

$$\vec{V}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ V_x & V_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (0-0)\hat{x} + (0-0)\hat{y} + (V_xB_y - V_yB_x)\hat{z}$$

$\vec{V}\times\vec{B} = (V_xB_y - V_yB_x)\hat{z}$. only $\hat{z}$ component

Therefore, $\vec{E}$ has only $\hat{z}$ component

(continued)

(2) prove $\vec{E} = E_z\hat{z}$ is a constant

From Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$

$\nabla \times \vec{E} = 0$, Assume steady state

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - 0\right)\hat{x} + \left(0 - \frac{\partial E_z}{\partial x}\right)\hat{y} + (0-0)\hat{z}$$

$$\Rightarrow \frac{\partial E_z}{\partial y}\hat{x} - \frac{\partial E_z}{\partial x}\hat{y} = 0$$

$$\Rightarrow \begin{cases} \dfrac{\partial E_z}{\partial y} = 0 & \Rightarrow E_z \text{ is a function independent of } y \\ \dfrac{\partial E_z}{\partial x} = 0 & \Rightarrow E_z \text{ is a function independent of } x \end{cases}$$

$\Rightarrow E_z$ is a constant across (x, y)

2. Stagnation - point flow model

(1) $L = \sqrt{\frac{2\eta a}{V_0}}$ $\eta = 1\ m^2 s^{-1}$, $a = 10000\ km = 10^7 m$

$V_0 = 10\ km/s = 10^4\ m/s$

$L = \sqrt{\frac{2 \cdot 1 \cdot 10^7 m}{10^4}} = \sqrt{2 \times 10^3}$

$\boxed{L = 44.7\ m}$: the half width of current sheet

(2) The general solution of this model is

$B_{(x)} = \frac{2Ea}{V_0 L} \operatorname{daw}\left(\frac{x}{L}\right)$

$\operatorname{daw}(x) = \exp(-x^2) \int_0^x \exp(t^2)\, dt$: the dawson function

Since we know at $x = L$, $B(L) = 1000\ G = 10^{-1}$ Tesla

$\operatorname{daw}(1) = e^{-1} \int_0^1 e^{t^2} dt = 0.54$

$E = \frac{B V_0 L}{2a} \frac{1}{\operatorname{daw}(1)} = \frac{10^{-1} \cdot 10^4 \cdot 44.7}{2 \cdot 10^7} \frac{1}{0.54} = 4.14 \times 10^{-3}\ V m^{-1}$

$E = 4.14 \times 10^{-3}\ V m^{-1}$ E is constant ~~everywhere~~

(3) $x = 0$

$\operatorname{daw}(0) = 0$ $\boxed{B(0) = 0}$

To obtain $\vec{j}$, use generalized ohm's Law

$\vec{j} = \sigma (\vec{E} + \vec{v} \times \vec{B})$

$\Rightarrow \boxed{j = \frac{1}{\mu_0 \eta} \left(E - \frac{V_0 x}{a} B\right)}$

$j = \frac{E}{\mu_0 \eta} = \frac{4.14 \times 10^{-3}}{4\pi \times 10^{-7} \cdot 1} = 3.29 \times 10^3\ A\ m^{-1}$

$j = 3.29 \times 10^3\ A\ m^{-2}$

~~Note: j is the density of current~~

~~$j = \frac{dB}{dt}$~~

(continued)

(4). $x = \frac{L}{2}$. within the current sheet

$$B(\tfrac{L}{2}) = \frac{2Ea}{v_0 L}\, daw(\tfrac{1}{2}) \quad , \quad daw(\tfrac{1}{2}) = 0.42$$

$$B(\tfrac{L}{2}) = \frac{2 \cdot 4.14\times10^{-3} \cdot 10^{7}}{10^{4} \cdot 44.7} \cdot 0.42 = 7.78\times10^{-2}\ T = 778\ G$$

$$j = \frac{1}{\mu_0 \eta}\left(E - \frac{v_0 x}{a} B\right)$$

$$j = \frac{1}{4\pi\times10^{-7} \cdot 1}\left(4.14\times10^{-3} - \frac{10^{4} \cdot \frac{44.7}{2}}{10^{7}} \cdot 7.78\times10^{-2}\right)$$

$$j = 1.97\times10^{3}\ A\ m^{-2}$$

$\boxed{B = 778\ G}$, $\boxed{j = 1.97\times10^{3}\ A\ m^{-2}}$

(5) $x = L$.

$daw(1) = 0.5381$

Similarly $\boxed{B = 1000\ G}$, $\boxed{j = -262\ A\ m^{-2}}$

(6) $x = 2L$

$daw(2) = 0.3013$

$\boxed{B = 558\ G}$ $\boxed{j = -675\ A\ m^{-2}}$

(7) $x = 10L$

$daw(10) = 0.0503$

$\boxed{B = 93.2\ G}$. $\boxed{j = -17\ A\ m^{-2}}$