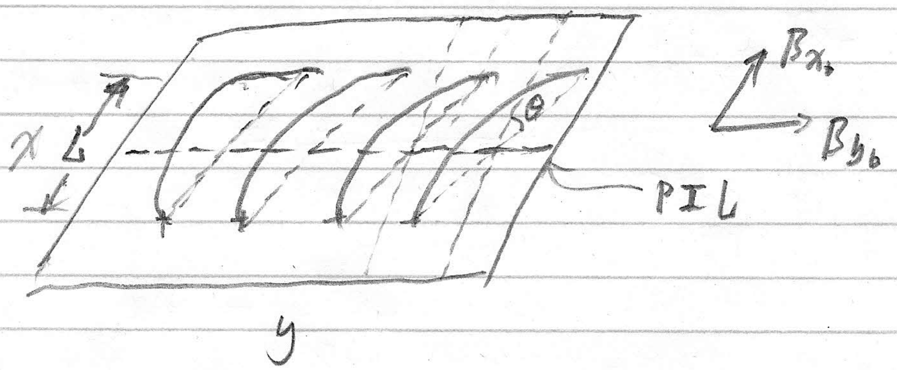
HW #3, Solution

1. Force free parameter α and sheared arcade (40 pts)



For sheared arcade solution, see Aschwanden CH 5.3.2

$$\begin{cases} B_{x0} = \frac{l}{k} B_0 & (1) \\ B_{y0} = \frac{\alpha}{k} B_0 & (2) \\ k^2 - l^2 - \alpha^2 = 0 & (3) \end{cases}$$

B_0: total magnetic field

B_{x_0}: strongest transverse $\vec{B} \perp PIL$

B_{y_0}: strongest ~~transverse~~ $\vec{B} \parallel PIL$

tilt angle: $\tan\theta = \frac{B_{y0}}{B_{x_0}} = \frac{\alpha}{l}$.

α: force free parameter $(\alpha = \frac{\mu_0 j}{B})$

l: characteristic size along z, $B_z \sim \exp(-lz)$

K: characteristic wave number: $B_x \sim \sin(kx)$

$k = \frac{\pi}{L}$. $B_x \sim \sin(\frac{\pi}{L}x)$

L: characteristic size of the loop arcade $\perp$ PIL

$B_x = 0$ at $x=0$, $x=L$

$B_0 = 1000\,G$, $L = 10^4$ km, $\theta = 30°$

(1) find B_{y0}

From (1), (2) and (3)

$$\left(\frac{B_{x0}}{B_0}\right)^2 + \left(\frac{B_{y0}}{B_0}\right)^2 = \frac{l^2}{k^2} + \frac{\alpha^2}{k^2} = \frac{l^2+\alpha^2}{k^2} = 1$$

(2)

HW#3 (continued)

$$B_{x_0}^2 + B_{y_0}^2 = B_0^2$$

$$B_{y_0} = B_0 \sin\theta = 1000 \cdot \sin 30^\circ = 500\,G$$

(2) Find B_{x_0} (note of the typo: in perpendicular to PIL)

$$B_{x_0} = B_0 \cos\theta = 1000 \cdot \cos 30^\circ \approx 867\ G$$

(3) ~~find~~ find α?

From (2): $\alpha = \frac{B_{y_0}}{B_0} \cdot k = \sin\theta \cdot \frac{2\pi}{L} = \frac{1}{2} \cdot \frac{2\pi}{10^4\,km \cdot 10^3 \frac{m}{km}}$

$$\alpha = 1.57 \times 10^{-7}\ m^{-1}$$

Note: the unit of α is $\underline{m^{-1}}$

(4) $\alpha = \frac{\mu_0 j}{B}$ (in MKS unit)

$$j = \frac{\alpha B}{\mu_0} = \frac{\frac{\pi}{2} \times 10^{-7} \cdot \left(1000\,G \cdot \frac{10^{-4}\,T}{1\,G}\right)}{4\pi \times 10^{-7}} \quad \text{in MKS or SI unit}$$

$$j = 1.25 \times 10^{-2}\ \underline{A\ m^{-2}} \qquad (j_1)$$

or:

$$\alpha = \frac{j}{B} \ \text{(in cgs unit)}$$

$$j = \alpha B = 1.57 \times 10^{-9}\ cm^{-1} \cdot 10^3\ G = 1.57 \times 10^{-6}\ G\ cm^{-1}$$

(in emu unit)

$$j = 1.57 \times 10^{-6} \cdot 3 \times 10^{10} = 4.71 \times 10^4 \ \text{statampere (in esu unit)}$$

Since $j(esu) / j(emu) = c = 3 \times 10^{10}$

$$j = 4.71 \times 10^4 \cdot \frac{1}{3} \cdot 10^{-5} = 1.57 \times 10^{-1}\ A\ m^{-2} \ \text{(in SI unit)}$$

" " Since 1 statampere $= \frac{1}{3} \times 10^{-5}\ A\ m^{-2}$ (j_2)

The "4π" difference between (j_1) and (j_2) is from "α" differ by 4π in $\left(\frac{SI}{cgs}\right)$

HW #3 Solution

2. magnetic null-point (40pts)

$$\vec{A} = (0, 0, A_z) = [0, 0, \frac{B_0}{2L_0}(y^2 - \cancel{\alpha}\, \bar{\alpha}^2 x^2)]$$

and $a = \bar{\alpha}^2$

(1) * $\vec{B} = \nabla \times \vec{A}$.

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & A_z \end{vmatrix} = \begin{pmatrix} \partial A_z/\partial y \\ -\partial A_z/\partial x \\ 0 \end{pmatrix} = \begin{pmatrix} B_0 \frac{y}{L_0} \\ B_0 a \frac{x}{L_0} \\ 0 \end{pmatrix}$$

$\vec{B}$ has only $\hat{x}, \hat{y}$ components. $B_x = B_0 \frac{y}{L_0}$, $B_y = B_0 a \frac{x}{L_0}$

* $\vec{j} = \frac{1}{\mu_0} \nabla \times \vec{B} = \frac{1}{\mu_0} \nabla \times (\nabla \times \vec{A}) = \frac{1}{\mu_0}[\nabla(\cancel{\nabla \cdot \vec{A}}) - \nabla^2 \vec{A}]$

$$\vec{j} = -\frac{1}{\mu_0} \nabla^2 \vec{A}$$

$$j_z = -\frac{1}{\mu_0}\left(\frac{d^2 A_z}{dx^2} + \frac{d^2 A_z}{dy^2}\right) = -\left(\frac{B_0}{\mu L_0}\right)(1-a)$$

$j_x = 0$

$j_y = 0$

* $$\vec{F} = \vec{j} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{j_x} & \cancel{j_y} & j_z \\ B_x & B_y & \cancel{B_z} \end{vmatrix} = \begin{pmatrix} -B_y j_z \\ B_x j_z \\ 0 \end{pmatrix}$$

$$\vec{F} = \begin{pmatrix} \frac{B_0^2 a(1-a)}{\mu L_0} \frac{x}{L_0} \\ -\frac{B_0^2 (1-a)}{\mu L_0} \frac{y}{L_0} \\ 0 \end{pmatrix} \begin{matrix} \leftarrow F_x \\ \leftarrow F_y \\ \leftarrow F_z \end{matrix}$$

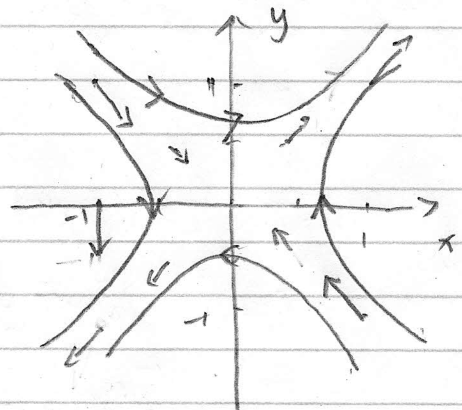
(continued)

(2) draw $\vec{B}$, $\vec{J}$, $\vec{F}$

$a = +1$, $B_x = y$, $B_y = x$ normalize to $\frac{B_0}{L_0}$

$j_z = 0$

$F_z = 0$



x - point configuration

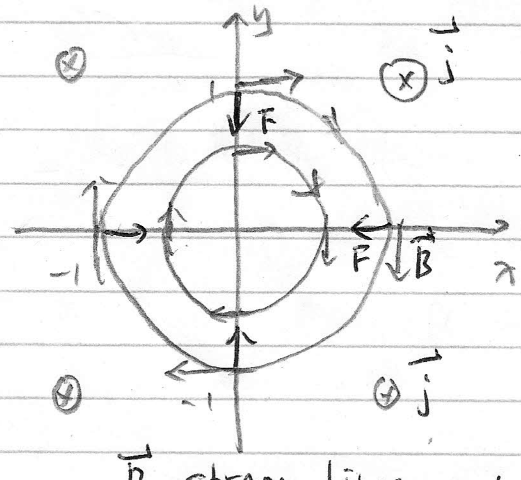
$\vec{B}$ stream lines and arrow ~~glphes~~ glyphs

$\vec{j} = 0$, $\vec{F} = 0$

$a = -1$ $B_x = y$, $B_y = -x$, normalize to $\frac{B_0}{L_0}$

$j_z = -\frac{B_0}{\mu L_0}$, constant, only $-\hat{z}$ direction

$F_x = x$, $F_y = -y$, normalize to $\frac{2B_0^2}{\mu L_0^2}$



O - point configuration

$\vec{B}$ stream lines, and arrow glyphs (continued).

(3) For $a = 1$, It is a force-free field

However, any pertubation will cause the configuration to collapse to form a current sheet.

because $\vec{F}$ will be along the direction of perturbation

For $a = -1$, It is not a force-free field.

O-point will collapse toward the center,

forming a singularity point with $\vec{j} \to \infty$

Hw #3 solution

3. Current sheet (20 pts)

$$B_y + iB_x = \frac{(z^2+1)}{(z^2+4)^{\frac{1}{2}}}$$

null point requires $B_x = 0$, $B_y = 0$

$$z^2 + 1 = 0 \qquad z^2 = -1$$

$$z = \pm i$$

current sheet: between the two singularity points

$$z^2 + 4 = 0$$

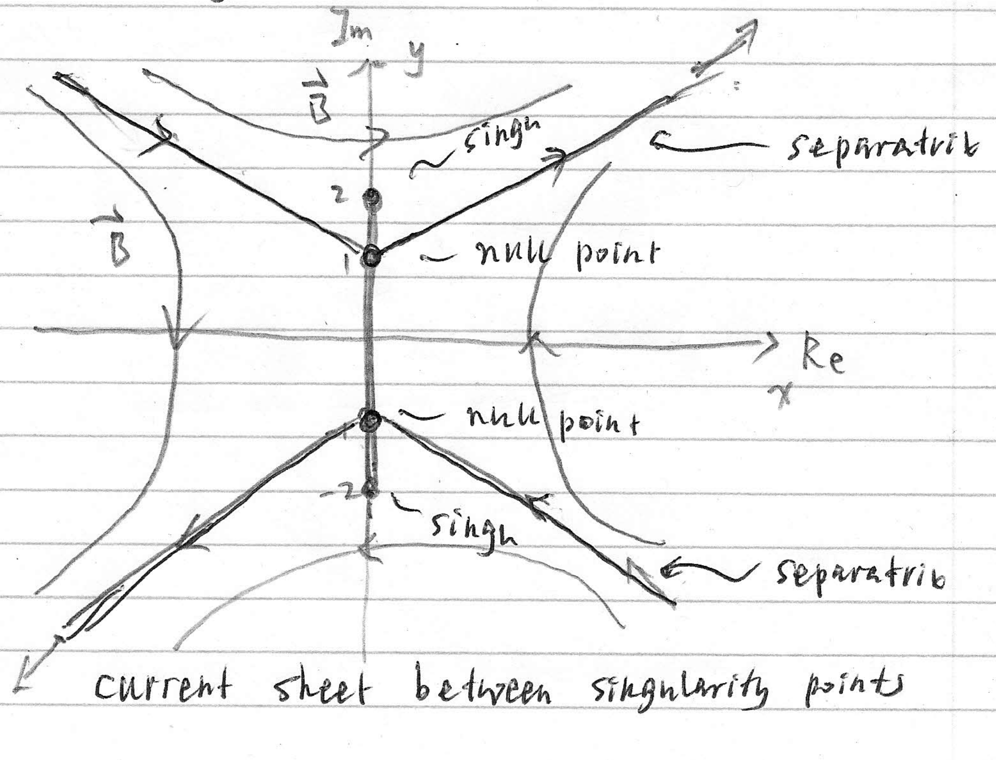
$$z^2 = -4$$

$$z = \pm 2i$$

In large distance $|z| \gg 1$

$$B_y + iB_x = z^2/z = z = x + iy$$

$$\begin{cases} B_x = y \\ B_y = x \end{cases}$$



current sheet between singularity points